Fredholm determinant of Hankel Composition Operators

Observe that the operators appearing in the Fredholm det of the above two examples

$K_t = M_t N_t$ Where $M_t, N_t : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ take the form

$$(M_t f)(x) := \int_0^\infty \phi_t(x+y) f(y)\, dy, \quad (N_t f)(x) := \int_0^\infty \psi_t(x+y) f(y)\, dy \qquad (*)$$

Where $\phi_t(x) := \phi(x+t)$, $\psi_t(x) := \psi(x+t)$, $\phi(x), \psi(x)$ functions decay fast at ∞.

GOE case : $\phi = \psi = Ai(x)$

GinOE case : $\phi = \psi = \frac{1}{\sqrt{\pi}} e^{-x^2}$

(Both cases assume equal ϕ & ψ & real-valued)

Before To get integrable Structure of K_t needs Contour integral reps / differential eqs for ϕ / ψ

(Krajenbrink 2020) Can treat $\det(1-K_t)$ with the form $(*)$ in a unified way

No DE/ Contour integral reps for ϕ/ψ needed $\quad \phi = \psi$ real-valued

(Bothner 2022+) Extension to the general case.

Thm1 (Krajenbrink 20', Bothner 22+')

Let $F(t) = \det(1-K_t)_{L^2(\mathbb{R}_+)}$ with K_t defined in $(*)$

For ϕ/ψ "nice" enough we have $\frac{d^2}{dt^2} \log F(t) = -q_0(t)\, q_0^*(t)$

Where $\begin{cases} q_0(t) = ((I-K_t)^{-1} \phi_t)(0) \\ q_0^*(t) = ((I-K_t^*)^{-1} \psi_t)(0) \end{cases}$

In the Self-adjoint case $\phi = \psi$ real-valued, $q_0 = q_0^*$ and $\frac{d^2}{dt^2} F(t) = -q_0(t)^2$

Remark 1° Thm 1 is purely a consequence of algebraic manipulations

2° Not much different from Tracy-Widom's original calculation

3° q_0 / q_0^* are still too implicit $\quad$ Aim: Better description of q_0/q_0^*!

To uncover more integrability we look at higher orders.

$$q_n(t) := \left((I-K_t)^{-1} D^n \phi_t \right)(0) \qquad p_n(t) := \operatorname{tr}\left((I-K_t)^{-1} D^n \phi_t \otimes \psi_t \right)$$

$$q_n^*(t) := \left((I-K_t^*)^{-1} D^n \psi_t \right)(0) \qquad p_n^*(t) := \operatorname{tr}\left((I-K_t^*)^{-1} D^n \phi_t \otimes \psi_t \right)$$

Then (p_n, p_n^*, q_n, q_n^*) satisfy the following peculiar ODE system

$$\begin{cases} \dfrac{dq_n}{dt} = q_{n+1} - q_0 p_n & \dfrac{dp_n}{dt} = -q_0^* q_n \\[2ex] \dfrac{dq_n^*}{dt} = q_{n+1}^* - q_0^* p_n^*(t) & \dfrac{dp_n^*}{dt} = -q_0 q_n^* \end{cases}$$

(Note that when $K_t = K_t^*$, $q_n = q_n^*$, $p_n = p_n^*$)

This system gives a lot of conservation laws, but it still does not give a closed differential equation for q_0 (As in the Airy case)

Nevertheless, this system is known to be a particular case of ZS system and

Ex 1° Airy $\quad X_+(z) = X_-(z) \begin{bmatrix} 1 & -ie^{-\frac{i}{3}z^3 - itz} \\ -ie^{\frac{i}{3}z^3 + itz} & 1 \end{bmatrix}$

$r_1(z) = \overline{r_2(\bar z)} = -ie^{\frac{i}{3}z^3} \qquad z \in \mathbb{R}$

Thm 2 (Krajenbrink 20', Bothner 22')

Given $t \in \mathbb{R}$ & ϕ, ψ. Let $X(z) = X(z; t, \phi, \psi) \in \mathbb{C}^{2\times 2}$ s.t

2° Gaussian $\quad r_1(z) = \overline{r_2(\bar z)} = -ie^{-\frac{1}{4}z^2} \qquad z \in \mathbb{R}$

$X_+(z) = X_-(z) \begin{bmatrix} 1-e^{-\frac{1}{2}z^2} & -ie^{-\frac{1}{4}z^2 - itz} \\ -ie^{-\frac{1}{4}z^2 + itz} & 1 \end{bmatrix} \qquad z \in \mathbb{R}$

1° $X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$

2° $X_\pm(z) := \lim_{\varepsilon \to 0^+} X(z \pm i\varepsilon)$ exists for $z \in \mathbb{R}$ and satisfy

$$X_+(z) = X_-(z) \begin{bmatrix} 1 - r_1(z) r_2(z) & -r_2(z) e^{-itz} \\ r_1(z) e^{itz} & 1 \end{bmatrix} \qquad z \in \mathbb{R}$$

With $r_1(z) = -i \int_{-\infty}^{\infty} \phi(y) e^{-izy}\, dy$ and $r_2(z) = i \int_{-\infty}^{\infty} \psi(y) e^{izy}\, dy$

3° $X(z) = I + X_1 z^{-1} + \cdots$ as $z \to \infty$ in $\mathbb{C}\setminus\mathbb{R}$ $\quad X_1 = X_1(t) = [X_1^{jk}(t)]_{j,k=1}^{2}$

Then $\quad X_1(t) = \begin{bmatrix} -ip_0(t) & q_0^*(t) \\ q_0(t) & ip_0^*(t) \end{bmatrix}$

In general $\quad X(t) = I + \sum_{k=1}^{N} z^{-k} \begin{bmatrix} (-i)^k p_{k-1} & i^{k-1} q_{k-1}^* \\ (-i)^{k-1} q_{k-1} & i^k p_{k-1}^* \end{bmatrix} + o(z^{-N})$ as $z \to +\infty$

Comments

1° Establishes a (one to one?) correspondence between

Fredholm det of Hankel composition Ops $\longleftrightarrow$ Zakharov - Shabat / AKNS Riemann-Hilbert prob

with kernel $K_t(x,y) = \int_0^\infty \phi_t(x+u)\psi_t(y+u)\,du$ with reflection coefficients given by Fourier transforms of ϕ/ψ

- The RHP representation gives useful information on the analytic/asymptotic info of $\det(I-K_t)$

- $\det(I-K_t)$ gives a (in some sense) explicit solution of the RHP, and hence some related differential equations. (Le Doussal - Krajenbrink 21')

Extensions

1° The kernels appearing in Hard edge scaling limits of RMT (Bessel, Meijer G-function, etc.) is usually of the form $K_t = t\int_0^1 \phi_t(xz)\psi_t(yz)\,dt$; $f_t(x) := f(tx)$

(multiplicative instead of additive)

Similar story, reflection coefficients given by Mellin transforms of ϕ/ψ

2° One can study the Hankel composition operator in weighted L^2 space of the form $K_t = M_t N_t$, $(M_t f)(x) = \int_0^\infty \phi_t(x+y) f(y)\sqrt{w(y)}\,dy$

$(N_t f)(x) = \int_0^\infty \psi_t(x+y) f(y)\sqrt{w(y)}\,dy$

$w : \mathbb{R} \to \mathbb{R}_{\geq 0}$ differentiable, $\nearrow$, bounded, $\int_{-\infty}^0 w(x)\,dx < \infty$.

Main example: Finite temperature Airy kernel (and its higher order analogue)

$$K_t^{\mathrm{Airy},\beta}(x,y) = \int_0^{+\infty} \mathrm{Ai}(x+u)\,\mathrm{Ai}(y+u)\cdot\frac{1}{1+e^{-\beta u}}\,du \qquad \beta>0.$$

$$\Rightarrow \frac{d^2}{dt^2}\log(\det(I-K_t)) = -\underbrace{\int_{\mathbb{R}} q_0(t,z)\,q_0^*(t,z)\,w(z)\,dz}$$

$$\begin{cases} q_0(t,z) := ((I-K_t)^{-1}\phi_{t+z})(0) & \text{nonlocal!} \\ q_0^*(t,z) := ((I-K_t^*)^{-1}\psi_{t+z})(0) \end{cases}$$

Derivation of the ODE systems

$$F(t) = \det(I - K_t)_{L^2(\mathbb{R}_+)}$$

$$q_n = (I-K_t)^{-1} D^n \phi_t\,(0) \qquad p_n = \mathrm{tr}((I-K_t)^{-1} D^n \phi_t \otimes \psi_t)$$

$$q_n^* = (I-K_t^*)^{-1} D^n \psi_t\,(0) \qquad p_n^* = \mathrm{tr}((I-K_t^*)^{-1} D^n \phi_t \otimes \psi_t)$$

Then $\frac{d^2}{dt^2}\log F(t) = -q_0(t)\, q_0^*(t)$

$$\begin{cases} \dfrac{dq_n}{dt} = q_{n+1} - q_0 p_n \quad (1.1) & \dfrac{dp_n}{dt} = -q_0^* q_n \quad (1.2) \\[2mm] \dfrac{dq_n^*}{dt} = q_{n+1}^* - q_0^* p_n^* \quad (1.1*) & \dfrac{dp_n^*}{dt} = -q_0 q_n^* \quad (1.2*) \end{cases}$$

Key observation

$$\frac{d}{dt} K_t(x,y) = \int_0^\infty \frac{d}{dt}\phi_t(x+u)\cdot \psi_t(y+u)\,du + \int_0^\infty \phi_t(x+u)\frac{d}{dt}\psi_t(y+u)\,du$$

$$= \int_0^\infty (D\phi)_t(x+u)\,\psi_t(y+u)\,du$$

$$= \phi_t(x+u)\psi_t(y+u)\Big|_0^{+\infty} = -\phi_t(x)\psi_t(y) \leftarrow \text{rank 1!}$$

$$\Rightarrow \frac{d}{dt}\log(\det(I-K_t)) = \frac{d}{dt}\mathrm{tr}(\log(I-K_t)) = \mathrm{tr}\left(-(I-K_t)^{-1}\frac{d}{dt}K_t\right)$$

$$= \mathrm{tr}((I-K_t)^{-1}\phi_t\otimes\psi_t) = p_0$$

This combined with (1.2) with $n=0$ shows that $\frac{d^2}{dt^2}\log(\det(I-K_t)) = -q_0 q_0^*$

We check (1.1) & (1.2)

(1.1)

$$\frac{dq_n}{dt} = \frac{d}{dt}(I-K_t)^{-1} D^n\phi_t(0) + (I-K_t)^{-1}\frac{d}{dt}D^n\phi_t(0)$$

$$= q_{n+1} + \underbrace{(I-K_t)^{-1}\frac{d}{dt}K_t\,(I-K_t)^{-1}}_{=\frac{d}{dt}(I-K_t)^{-1}} D^n\phi_t(0)$$

$$= q_{n+1} + (I-K_t)^{-1}(-\phi_t\otimes\psi_t)(I-K_t)^{-1}D^n\phi_t(0)$$

$$= q_{n+1} - [(I-K_t)^{-1}\phi_t]\otimes[(I-K_t^*)^{-1}\psi_t]\,D^n\phi_t(0)$$

$$= q_{n+1} - (I-K_t)^{-1}\phi_t(0)\cdot \mathrm{tr}((I-K_t^*)^{-1}\psi_t\otimes D^n\phi_t)$$

$$= q_{n+1} - q_0\cdot p_n$$

(1.2) A repeated use of integration by parts ...

Introducing the differential operator $D := \frac{d}{dx}$ $\quad (Df(x) = f'(x))$

Then $\frac{d}{dt}\phi_t = (D\phi)_t$

$$\Rightarrow \frac{dP_n}{dt} = \underbrace{\mathrm{tr}((I-K_t)^{-1} D^{n+1}\phi_t \otimes \psi_t)}_{\mathrm{I}} + \underbrace{\mathrm{tr}((I-K_t) D^n\phi_t \otimes D\psi_t)}_{\mathrm{II}}$$
$$+ \underbrace{\mathrm{tr}\left(\frac{d}{dt}(I-K_t)^{-1} D^n\phi_t \otimes \psi_t\right)}_{\mathrm{III}}$$

For III, Use 1) $\frac{d}{dt}(I-K_t)^{-1} = (I-K_t)^{-1}\frac{d}{dt}K_t\,(I-K_t)^{-1}$

2) $\frac{d}{dt}K_t = DK_t - K_tD - K_t\Delta_0$, where $\Delta_0 f(x) = f(x)\delta_0(x)$

2) is a repeated use of IBP, namely

$$DK_tf(x) = \frac{\partial}{\partial x}(K_tf(x)) = \int_0^\infty dy\int_0^\infty d\lambda\, D\phi_t(x+\lambda)\psi_t(y+\lambda)f(y)$$
$$= \int_0^\infty f(y)\,dy\left(-\phi_t(x)\psi_t(y) - \int_0^\infty d\lambda\,\phi_t(x+\lambda)D\psi_t(y+\lambda)\right)$$
$$= \frac{d}{dt}K_t f(x) - \int_0^\infty dy\int_0^\infty d\lambda\,\phi_t(x+\lambda)D\psi_t(y+\lambda)f(y)$$
$$= \frac{d}{dt}K_t f(x) + K_t(x,0)f(0) + K_tDf(x)$$

3) $[D,(I-K_t)^{-1}] = (I-K_t)^{-1}[D,K_t](I-K_t)^{-1}$

By 1),2),3), $\mathrm{III} = \mathrm{tr}((I-K_t)^{-1}(DK_t - K_tD - K_t\Delta_0)(I-K_t)^{-1}D^n\phi_t\otimes\psi_t)$

$$= \underbrace{\mathrm{tr}([D,(I-K_t)^{-1}]D^n\phi_t\otimes\psi_t)}_{-\mathrm{II}-(I-K_t)^{-1}\phi_t(0)\psi_t(0)} - \mathrm{tr}((I-K_t)^{-1}K_t\Delta_0(I-K_t)^{-1}D^n\phi_t\otimes\psi_t)$$
$$= \mathrm{tr}(D(I-K_t)^{-1}D^n\phi_t\otimes\psi_t) - \underbrace{\mathrm{tr}((I-K_t)^{-1}D^{n+1}\phi_t\otimes\psi_t)}_{\mathrm{I}}$$
$$- \mathrm{tr}((I-K_t)^{-1}K_t\Delta_0(I-K_t)^{-1}D^n\phi_t\otimes\psi_t)$$
$$= -\mathrm{I} - \mathrm{II} - \underbrace{(I-K_t)^{-1}D^n\phi_t(0)\cdot\psi_t(0)}_{= \mathrm{tr}((I-K_t)^{-1}(I-K_t)\delta_0\otimes\psi_t)}$$
$$- (I-K_t)^{-1}D^n\phi_t(0)\cdot\mathrm{tr}((I-K_t)^{-1}K_t\delta_0\otimes\psi_t)$$
$$= -\mathrm{I} - \mathrm{II} - (I-K_t)^{-1}D^n\phi_t(0)\cdot\mathrm{tr}((I-K_t)^{-1}\delta_0\otimes\psi_t)$$
$$= -\mathrm{I} - \mathrm{II} - (I-K_t)^{-1}D^n\phi_t(0)\cdot(I-K_t^*)^{-1}\psi_t(0)$$

▨

How the ODE system arises from the RHP? $J(z;t)$

Recall $X_+(z) = X_-(z) \cdot \begin{bmatrix} 1 - r_1(z) r_2(z) & -r_2(z) \underbrace{e^{-itz}}_{} \\ r_1(z) e^{itz} & 1 \end{bmatrix}$ the only term depending on t $z \in \mathbb{R}$

$$\Psi(z) := X(z) \cdot e^{-\frac{itz}{2}\sigma_3} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfies $\Psi_+(z) = X_+(z) \cdot e^{-\frac{itz\sigma_3}{2}} = X_-(z) \cdot J(z;t)\, e^{-\frac{itz}{2}\sigma_3}$

$$= \Psi_-(z)\, e^{\frac{itz}{2}\sigma_3} J(z;t)\, e^{-\frac{itz}{2}\sigma_3}$$

$$e^{\frac{itz}{2}\sigma_3} J(z;t)\, e^{-\frac{itz}{2}\sigma_3} = \underbrace{\hat{J}(z)}_{\text{Independent of } t!} = \begin{bmatrix} 1 - r_1(z) r_2(z) & -r_2(z) \\ r_1(z) & 1 \end{bmatrix}$$

$$\Rightarrow \quad \Psi_+(z) = \Psi_-(z)\, \hat{J}(z)$$

$$\Rightarrow \quad \left(\frac{\partial \Psi}{\partial t}\right)_+(z) = \left(\frac{\partial \Psi}{\partial t}\right)_-(z)\, \hat{J}(z)$$

$\Rightarrow \quad \frac{\partial \Psi}{\partial t} \Psi^{-1}$ no jump on $\mathbb{R}$ $\Rightarrow$ entire function

$X(z) \sim I + X_1 z^{-1} + O(z^{-2})$ as $z \to +\infty$

$(I + X_1 z^{-1} + O(z^{-2})) \cdot e^{-\frac{itz}{2}\sigma_3}$

$(I - X_1 z^{-1} + O(z^{-2}))(e^{+\frac{itz}{2}\sigma_3})$

$$\Psi(z) \sim e^{-\frac{itz}{2}\sigma_3} + X_1 e^{-\frac{itz}{2}\sigma_3} z^{-1} + O(z^{-2})$$

$$\frac{\partial \Psi}{\partial t}(z) \sim -\frac{iz}{2}\sigma_3 e^{-\frac{itz}{2}\sigma_3} + X_1 e^{-\frac{itz}{2}\sigma_3} \cdot \left(-\frac{iz}{2}\sigma_3\right) z^{-1} + O(z^{-1})$$

$$\frac{\partial \Psi}{\partial t} \Psi^{-1} \sim -\frac{iz}{2}\sigma_3 + \left[\frac{i\sigma_3}{2}, X_1\right] + O(z^{-1}) \quad \text{as } z \to +\infty$$

$$\Rightarrow \quad \frac{\partial \Psi}{\partial t} = \left(-\frac{iz}{2}\sigma_3 + \frac{i}{2}[\sigma_3, X_1]\right)\Psi \quad \leftarrow \text{Zakharov-Shabat}$$

$$\Rightarrow \quad \frac{\partial X}{\partial t} = \left(-\frac{iz}{2}[\sigma_3, X] + i\begin{bmatrix} 0 & q_0^* \\ -q_0 & 0 \end{bmatrix}\right) X \qquad (*)$$

Write $X = I + \sum_{k=1}^{\infty} z^{-k} X_k$ (formally)

Look at coeff of z^{-k} at both sides of $(*)$

$$\Rightarrow \frac{\partial X_k}{\partial t} = -\frac{i}{2}[\sigma_3, X_{k+1}] + i \begin{bmatrix} 0 & q_0^* \\ -q_0 & 0 \end{bmatrix} X_k$$

$$\Rightarrow \text{If } X_k = \begin{bmatrix} (-i)^k p_{k-1} & i^{k-1} q_{k-1}^* \\ (-i)^{k-1} q_{k-1} & i^k p_{k-1}^* \end{bmatrix}$$

Then $\frac{\partial p_{k-1}}{\partial t} = -q_0^* q_{k-1}$ etc.

Combine the representation of p_n's & q_n's we can formally write

(recall that $\begin{cases} p_n = tr((I-K_t)^{-1} D^n \phi_t \otimes N_t \delta_0 \\ q_n = tr((I-K_t)^{-1} D^n \phi_t \otimes \delta_0 \end{cases}$ ...)

$$X(z) = I + \sum_{l=1}^{\infty} z^{-l} \begin{bmatrix} (-i)^l p_{l-1} & i^{l-1} q_{l-1}^* \\ (-i)^{l-1} q_{l-1} & i^l p_{l-1}^* \end{bmatrix}$$

(The signs could be wrong...)

$$= I + tr_{L^2(\mathbb{R}_+)}\left(\delta_0 \otimes \begin{bmatrix} -N_t (I-K_t)^{-1}(D-zi)^{-1} & i(I-K_t^*)^{-1}(D+zi)^{-1} \\ -i(I-K_t)^{-1}(D-zi)^{-1} & -N_t(I-K_t^*)^{-1}(D+zi)^{-1} \end{bmatrix} M_t \delta_0 \right) \quad (**)$$

This provides a candidate of the solution of the RHP !

Need to show $(**)$ satisfies (i) analyticity (ii) Jump condition (iii) Normalization

Key property to use

The resolvent of differential operator $(D+zi)^{-1}$ can be written as

$$(D+iz)^{-1} g(x) = \begin{cases} -\int_0^{\infty} dt\, g(x+t) e^{izt} & \text{for } Im(z) > 0 \\ \int_{-\infty}^{0} dt\, g(x+t) e^{izt} & \text{for } Im(z) < 0 \end{cases}$$