Fredholm determinants / Pfaffians and Integrable differential equations

Consider the Gaussian Orthogonal Ensemble (GOE) $X = \frac{1}{\sqrt{2}}(Y + Y^T) \in \mathbb{R}^{n\times n}$, Y_{ij} iid $N(0,1)$

Thm 1.1 (Moore 1990, Forrester 1992)

$$\max_{1\le i\le n} \lambda_i(X) \sim 2\sqrt{n} + \frac{\chi_1}{n^{1/6}}$$

Where $\underset{F_1(t)}{\underset{\shortparallel}{\mathbb{P}(\chi_1 \le t)}} \text{"="} \sqrt{\det(I - K_{Airy} - a \otimes A)_{L^2(t,+\infty)}}$ (1.1)

$$K_{Airy}(x,y) = \int_0^{+\infty} Ai(x+u)Ai(y+u)\,du$$ — Airy kernel $= \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$

$a(x) = Ai(x)$ $A(x) = 1 - \int_0^{+\infty} Ai(x+u)\,du$ $1 - \langle (I - K_{Airy})^{-1} a, A\rangle$

A bit more carefully, $\det(I - K_{Airy} - a\otimes A) = \det(I - K_{Airy}) \cdot \underset{\shortparallel}{\det(I - (I - K_{Airy})^{-1} a \otimes A)}$

An alternative representation

Thm 1.2 (Tracy-Widom 1993, 1994)

$$F_1(t) = \exp\left[-\frac{1}{2}\int_t^\infty (s-t)(q(s))^2\,ds - \frac{1}{2}\int_t^\infty q(s)\,ds\right] \quad (1.2)$$

Where $q(s)$ is the unique solution to the ODE

$$q'' = sq + 2q^3 \quad \text{s.t} \quad q(s) \sim Ai(s) \text{ as } s \to +\infty$$

(Painlevé II Hastings-Mcleod solution)

Main goal: Understand the relationship between (1.1) and (1.2)

The kernel K_{Airy} is of the form $\frac{\vec{f}^t(x)\vec{g}(y)}{x-y}$ for $f(x)$, $g(y)$ k-dim column vector with $\langle f(x), g(x)\rangle = 0$

Such integral operators are usually called (IIKS Its-Izergin-Korepin-Slavnov) integrable op

(Typical examples include Christoffel-Darboux kernel of OPs)

Key properties of Integrable operators

$(I-K)^{-1} - I$ is also integrable with kernel $R(x,y) = \frac{\vec{F}(x)^T \vec{G}(y)}{x-y}$, where $\vec{F} = (I-K)^{-1}\vec{f}$, $\vec{G} = (I-K)^{-1}\vec{g}$

Moreover $\vec{F}, \vec{G}$ are expressed as $M_\pm f$, $(M_\pm^{-1})^T g$ for $k\times k$ matrix valued function M satisfying certain B.V.P (known as a Riemann-Hilbert problem RHP)

There is a parallel story ...

Consider $Y \in \mathbb{R}^{n\times n}$, Y_{ij} i.i.d $N(0,1)$ Real Ginibre ensemble GinOE

Thm 2.1 (Rider - Sinclair 14', Poplavskyi - Tribe - Zaboronski 16')

$$\max_{\substack{1\le i\le n \\ \lambda_i \in \mathbb{R}}} \lambda_i(Y) \sim \sqrt{n} + \chi$$

$$\mathbb{P}(\chi \le t) \text{ "=" } \sqrt{\det(I - K_{\text{Gaussian}} - g\otimes G)_{L^2(t,+\infty)}}$$

Where
$$g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}, \quad G(x) = \int_{-\infty}^{x} g(u)\,du$$
$$K_{\text{Gaussian}}(x,y) = \frac{1}{\pi}\int_0^{+\infty} e^{-(x+u)^2} e^{-(y+u)^2}\,du$$

K_{Gaussian} is not of integrable type (Although it is conjugate to an integrable op in Fourier space)

Thm 2.2 (Baik - Bothner 18')

$$\mathbb{P}(\chi \le t) = \exp\left[-\frac{1}{2}\int_t^{\infty}(s-t)\,p(s)^2\,ds - \frac{1}{2}\int_t^{\infty} p(s)\,ds\right]$$

Where $p(s) = X_1^{12}(\frac{s}{2})$, related to the (12)-th entry of a $\mathbb{C}^{2\times 2}$ matrix-valued function X_1 determined as follows:

(RHP for ZS - AKNS System) For $x \in \mathbb{R}$, determine $X(z) = X(z;x) \in \mathbb{C}^{2\times 2}$ s.t

(1) (Analyticity) $X(z)$ is analytic for $z \in \mathbb{C}\setminus\mathbb{R}$ and extends continuously to upper/lower half-plane.

(2) (Jump) $$X_+(z) = X_-(z)\begin{bmatrix} 1-|r(z)|^2 & -r(z)^* e^{-2ixz} \\ r(z)e^{2ixz} & 1 \end{bmatrix}, \quad z\in\mathbb{R}$$
$$r(z) = -ie^{-\frac{1}{4}z^2}$$
$$X_\pm(z) = \lim_{\varepsilon\to 0+} X(z \pm i\varepsilon)$$

(3) (Normalization) $$X(z) = I + X_1 z^{-1} + X_2 z^{-2} + \cdots \quad \text{as } z\to\infty$$

$$X_i = X_i(x) = [X_i^{jk}(x)]_{j,k=1}^{2}$$

Remark Let $\Psi(z) := X(z)e^{-ixz\sigma_3}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}$, Then $\Psi_+(z) = \Psi_-(z)\begin{bmatrix} 1-|r(z)|^2 & -r(z)^* \\ r(z) & 1\end{bmatrix}$ for $z\in\mathbb{R}$

independent of x!

$\Rightarrow$ $\frac{\partial \Psi}{\partial x}\Psi^{-1}$ entire function in z, condition (3) + Liouville's thm

$$\Rightarrow \frac{\partial \Psi}{\partial x} = \left(-iz\sigma_3 + \begin{bmatrix} 0 & 2ip \\ (2ip)^* & 0\end{bmatrix}\right)\Psi$$

Historically this is what is called a Zakharov - Shabat System

As a consequence $$\frac{\partial X}{\partial x} = iz[X,\sigma_3] + 2i\begin{bmatrix} 0 & p \\ -p & 0\end{bmatrix}X$$

A Quick overview of Inverse Scattering Method

The Zakharov-Shabat system arises in the study of (defocusing) Nonlinear Schrödinger Eq

$$i u_t + \frac{1}{2} u_{tt} - |u|^2 u = 0 \quad (*) \quad \text{for complex-valued } u(x,t)$$

In fact $(*)$ can be viewed as the compatibility condition for the simultaneous linear eq of Lax pair

$$\begin{cases} \dfrac{\partial \Psi}{\partial x} = U \Psi \\[2ex] \dfrac{\partial \Psi}{\partial t} = V \Psi \end{cases} \qquad U = U(x,t;\lambda) := \begin{bmatrix} -i\lambda & u \\ u^* & i\lambda \end{bmatrix}$$

$$V = V(x,t;\lambda) := \begin{bmatrix} -i\lambda^2 - i\frac{1}{2}|u|^2 & \lambda u + i\frac{1}{2} u_x \\ \lambda u^* - i\frac{1}{2} u^*_x & i\lambda^2 + i\frac{1}{2}|u|^2 \end{bmatrix}$$

$$\left(\frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial t}\right) \right) \Rightarrow \frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0 \leftarrow \text{(defocusing) NLS}$$

The spatial part $\frac{\partial \Psi}{\partial x} = U\Psi$ is precisely the ZS system.

To solve I.V.P for linear Schrödinger Eq $\begin{cases} i u_t + \frac{1}{2} u_{xx} = 0 \\ u(x,0) = u_0(x) \end{cases}$ we use Fourier transform method

$$\hat{u}(\lambda,t) := \int_{\mathbb{R}} u(x,t) e^{2i\lambda x} \, dx \Rightarrow \begin{cases} i\hat{u}_t - 2\lambda^2 \hat{u} = 0 \\ \hat{u}(\lambda,0) = \hat{u}_0(\lambda) \end{cases} \Rightarrow \hat{u}(\lambda,t) = e^{-2i\lambda^2 t} \hat{u}(\lambda,0)$$

Finally $u(x,t) = \frac{1}{\pi} \int_{\mathbb{R}} \hat{u}(\lambda,t) e^{-2i\lambda x} \, d\lambda$

Graphically

$$\begin{tikzcd} u_0(x) \arrow[r, dashed, "\text{I.V.P for linear Schrödinger}"] \arrow[d, "\text{Direct Fourier}"'] & u(x,t) \\ \hat{u}_0(\lambda) \arrow[r, "\text{time evolution}"] & \hat{u}(\lambda,t) = \hat{u}_0(\lambda) \cdot e^{-2i\lambda^2 t} \arrow[u, "\text{Inverse Fourier}"'] \end{tikzcd}$$

There is an analogy for the nonlinear problem

Consider $\begin{cases} i u_t + \frac{1}{2} u_{xx} - |u|^2 u = 0 \\ u(x,0) = u_0(x) \in S(\mathbb{R}) \end{cases}$

$$\begin{tikzcd} u_0(x) \arrow[r, dashed, "\text{I.V.P for (defocusing) NLS}"] \arrow[d, "\text{Direct Scattering transform}"'] & u(x,t) \\ R(\lambda) \arrow[r, "\text{time evolution}"] & R(\lambda,t) = R(\lambda) e^{2i\lambda^2 t} \arrow[u, "\text{Inverse Scattering transform}"'] \end{tikzcd}$$

reflection coefficients

Caveats

1° The direct scattering transform involves an eigenvalue problem of the form

$$\left(i\sigma_3 \frac{\partial}{\partial x} + \begin{bmatrix} 0 & -iu \\ iu^* & 0 \end{bmatrix} \right) \Psi = \lambda \Psi$$

2° The inverse scattering transform can be formulated as a Riemann-Hilbert B.V.P (At least in 1 spatial dimension)

3° Both are linear! (Usually not explicitly solvable, but neither is the Fourier!)

4° Useful for analytic/asymptotic properties (Nonlinear steepest descent method)

Fredholm determinant of Hankel Composition Operators

Observe that the operators appearing in the Fredholm det of the above two examples

$K_t = M_t N_t$ Where $M_t, N_t : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ take the form

$$(M_t f)(x) := \int_0^\infty \phi_t(x+y) f(y)\, dy \, , \quad (N_t f)(x) := \int_0^\infty \psi_t(x+y) f(y)\, dy \qquad (*)$$

Where $\phi_t(x) := \phi(x+t)$, $\psi_t(x) := \psi(x+t)$, $\phi(x), \psi(x)$ functions decay fast at ∞.

GOE case : $\phi = \psi = Ai(x)$

GinOE case : $\phi = \psi = \frac{1}{\sqrt{\pi}} e^{-x^2}$

(Both cases assume equal ϕ & ψ & real-valued)

Before To get integrable Structure of K_t needs Contour integral reps / differential eqs for ϕ / ψ

(Krajenbrink 2020) Can treat $\det(1-K_t)$ with the form $(*)$ in a unified way

No DE / Contour integral reps for ϕ/ψ needed $\phi = \psi$ real-valued

(Bothner 2022+) Extension to the general case.

Thm1(Krajenbrink 20', Bothner 22+')

Let $F(t) = \det(1-K_t)_{L^2(\mathbb{R}_+)}$ with K_t defined in $(*)$

For ϕ/ψ "nice" enough we have $\frac{d^2}{dt^2} \log F(t) = -q_0(t)\, q_0^*(t)$

Where
$$\begin{cases} q_0(t) = ((I-K_t)^{-1}\phi_t)(0) \\ q_0^*(t) = ((I-K_t^*)^{-1}\psi_t)(0) \end{cases}$$

In the self-adjoint case $\phi = \psi$ real-valued, $q_0 = q_0^*$ and $\frac{d^2}{dt^2} F(t) = -q_0(t)^2$

Remark 1° Thm 1 is purely a consequence of algebraic manipulations

2° Not much different from Tracy-Widom's original calculation

3° q_0 / q_0^* are still too implicit Aim: Better description of q_0 / q_0^*!

To uncover more integrability we look at higher orders.

$$q_n(t) := \left((I-K_t)^{-1} D^n \phi_t \right)(0) \qquad p_n(t) := \mathrm{tr}\left((I-K_t)^{-1} D^n \phi_t \otimes \psi_t \right)$$

$$q_n^*(t) := \left((I-K_t^*)^{-1} D^n \psi_t \right)(0) \qquad p_n^*(t) := \mathrm{tr}\left((I-K_t^*)^{-1} D^n \phi_t \otimes \psi_t \right)$$

Then (p_n, p_n^*, q_n, q_n^*) satisfy the following peculiar ODE system

$$\begin{cases} \dfrac{dq_n}{dt} = q_{n+1} - q_0 p_n & \dfrac{dp_n}{dt} = -q_0^* q_n \\ \dfrac{dq_n^*}{dt} = q_{n+1}^* - q_0^* p_n^*(t) & \dfrac{dp_n^*}{dt} = -q_0 q_n^* \end{cases}$$

(Note that when $K_t = K_t^*$, $q_n = q_n^*$, $p_n = p_n^*$)

This system gives a lot of conservation laws, but it still does not give a closed differential equation for q_0 (As in the Airy case)

Nevertheless, this system is known to be a particular case of ZS system and

Thm 2 (Krajenbrink 20', Bothner 22')

Given $t \in \mathbb{R}$ & ϕ, ψ. Let $X(z) = X(z; t, \phi, \psi) \in \mathbb{C}^{2\times 2}$ s.t

1° $X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$

2° $X_\pm(z) := \lim_{\varepsilon \to 0^+} X(z \pm i\varepsilon)$ exists for $z \in \mathbb{R}$ and satisfy

$$X_+(z) = X_-(z) \begin{bmatrix} 1 - r_1(z) r_2(z) & -r_2(z) e^{-itz} \\ r_1(z) e^{itz} & 1 \end{bmatrix} \qquad z \in \mathbb{R}$$

With $r_1(z) = -i \int_{-\infty}^{\infty} \phi(y) e^{-izy} dy$ and $r_2(z) = i \int_{-\infty}^{\infty} \psi(y) e^{izy} dy$

3° $X(z) = I + X_1 z^{-1} + \cdots$ as $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$ $X_1 = X_1(t) = [X_1^{jk}(t)]_{j,k=1}^{2}$

Then $$X_1(t) = \begin{bmatrix} -i p_0(t) & q_0^*(t) \\ q_0(t) & i p_0^*(t) \end{bmatrix}$$

In general $$X(t) = I + \sum_{k=1}^{\infty} z^{-k} \begin{bmatrix} (-i)^k p_{k-1} & i^{k-1} q_{k-1}^* \\ (-i)^{k-1} q_{k-1} & i^k p_{k-1}^* \end{bmatrix} \quad \text{as } z \to +\infty$$

If ϕ, ψ smooth.

Comments

1° Establishes a (one to one?) correspondence between

Fredholm det of Hankel composition Ops $\leftrightarrow$ Zakharov - Shabat / AKNS Riemann-Hilbert prob

with kernel $K_t(x,y) = \int_0^\infty \phi_t(x+u)\psi_t(y+u)\,du$ with reflection coefficients given by Fourier transforms of ϕ/ψ

- The RHP representation gives useful information on the analytic/asymptotic info of $\det(I-K_t)$

- $\det(I-K_t)$ gives a (in some sense) explicit solution of the RHP, and hence some related differential equations. (Le Doussal - Krajenbrink 21')

Extensions

1° The kernels appearing in Hard edge scaling limits of RMT (Bessel, Meijer G-function, etc.) is usually of the form $K_t = t\int_0^1 \phi_t(xz)\psi_t(yz)\,dt$; $f_t(x) := f(tx)$

(multiplicative instead of additive)

Similar story, reflection coefficients given by Mellin transforms of ϕ/ψ

2° One can study the Hankel composition operator in weighted L^2 space of the form $K_t = M_t N_t$,

$$(M_t f)(x) = \int_0^\infty \phi_t(x+y) f(y) \sqrt{w(y)}\,dy$$

$$(N_t f)(x) = \int_0^\infty \psi_t(x+y) f(y) \sqrt{w(y)}\,dy$$

$w: \mathbb{R} \to \mathbb{R}_{\geq 0}$ differentiable, $\nearrow$, bounded, $\int_{-\infty}^0 w(x)\,dx < \infty$.

Main example: Finite temperature Airy kernel (and its higher order analogue)

$$K_t^{\mathrm{Airy},\beta}(x,y) = \int_0^{+\infty} \mathrm{Ai}(x+u)\,\mathrm{Ai}(y+u)\cdot \frac{1}{1+e^{-\beta u}}\,du \qquad \beta > 0.$$

$$\Rightarrow \frac{d^2}{dt^2}\log(\det(I-K_t)) = -\underbrace{\int_{\mathbb{R}} q_0(t,z)\, q_0^*(t,z)\, w(z)\,dz}_{\text{nonlocal!}}$$

$$\begin{cases} q_0(t,z) := ((I-K_t)^{-1}\phi_{t+z})(0) \\ q_0^*(t,z) := ((I-K_t^*)^{-1}\psi_{t+z})(0) \end{cases}$$