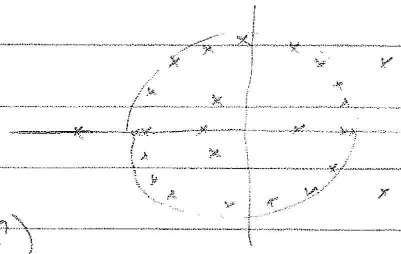
Tu

(0) DEMBO / ZEITOUNI / POONEN / SHAO , 2002

$$f(x) = \sum_{K=0}^{2N} A_K x^K \qquad A_K \text{ IID}, \ \mathbb{E}[A_K]=0, \ \mathbb{E}[A_K^2]=1$$

(logarithmic)

$$P[f(x)>0 \text{ on } \mathbb{R}] \approx N^{-3/4}$$



$(A_K) \sim N(0,1) \Rightarrow$ Real zeros are Pfaffian P.P

GAP PROB. – ASYMPTOTICS (TI ?)

(1) X simple P.P on $\mathbb{R}$, $\rho(x_1 \dots x_M)$ intensities $= \begin{cases} \mathrm{Det}(K(x_i,x_j) : i,j \le M) & \mathrm{Det}\ \rho \\ \mathrm{Pf}(\mathbb{K}(x_i,x_j) : i,j \le M) & \mathrm{Pf}\ \rho \end{cases}$

$K : \mathbb{R}\times\mathbb{R} \to \mathbb{R} \qquad \mathbb{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$

$$P[X[a,b] \text{ is empty}] = \sum_{M=0}^{\infty} \frac{(-1)^M}{M!} \int_{[a,b]^M} \rho(x_1 \dots x_M)\, d\underline{x} \ \overset{?}{\Longleftarrow} \begin{cases} \mathrm{Det}_{[a,b]}(I-K) \\ \mathrm{Pf}_{[a,b]}(J-\mathbb{K}) \end{cases}$$

Thinning X $\begin{cases} \text{leave prob } p \\ \text{remove prob } 1-p \end{cases} \to X_p$ — kernel pK / $p\mathbb{K}$.

$p \to P[X_p[a,b]=0]$ decreasing with $p \in [0,1]$

(2) Plemelj – Smithies formula.

$$\ln \mathrm{Det}(I-pK) = -\sum_{M=1}^{\infty} \frac{p^M}{M} \mathrm{Tr}(K^M)$$

Always True for small $|p|$

$$\ln\Big(\prod_i (1-p\lambda_i)\Big) = \sum_i \ln(1-p\lambda_i) = -\sum_K \sum_i \frac{p^K}{K} \lambda_i^K$$

Need: $|p\lambda_i| <$

$\lambda_i \to 0$ always

- $K : L^2[a,b] \to L^2[a,b]$ $\quad Kf(x) = \int_a^b K(x,y) f(y)\, dy$
- $\mathbb{K}^2 (L^2[a,b])^2 \to (L^2[a,b])^2$

(3) T-I case $K(x,y) = f(y-x)$. $\int_{\mathbb{R}} |f(z)|\, dz < \infty$.

$$\mathrm{Tr}(K^M) = \int_{[0,L]^M} f(x_2-x_1) f(x_3-x_2) \dots f(x_M - x_{M-1}) f(x_1 - x_M)\, d\underline{x}$$

$\begin{cases} x_1 = y_1 \\ x_2 = y_2 + y \\ \vdots \\ x_N = y_N + \dots \end{cases}$

$$\approx \int_{[0,L]} dy_1 \int_{\boxed{/\!/}} f(y_2) f(y_3) \dots f(y_N) f(-y_2-y_3-\dots-y_N)\, dy_2 \dots dy_M$$

$\boxed{/\!/}$: $-y_1 \le y_2 \le L-y_1$ ...

$\boxed{/\!/} \approx \mathbb{R}^{M-1}$

Tue 2

$$= L \int_{\mathbb{R}^M} f(y_1)\, f(y_2)\, f(y_3) \dots f(y_M)\, \delta_0(y_1 + \dots + y_M)\, dy + o(L)$$

$$= L \int d\theta\, e^{i\theta(y_1 + \dots + y_M)} f(y_1) \dots f(y_M)\, dy + o(L) \qquad \cancel{\tfrac{1}{2\pi}}$$

$$= L \int d\theta\, (\hat{f}(\theta))^M + o(L)$$

Now

$$\log \mathrm{Det}(I - pK) = -\sum_M \frac{p^M}{M} \int d\theta\, \hat{f}(\theta)^M \cdot L + o(L)$$

$$= \int \ln(1 - \hat{f}(\theta))\, d\theta \cdot L + o(L).$$

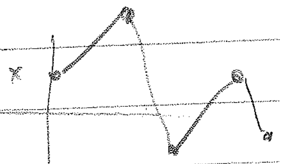
Kac Correct ✓ and improved to $= c_1(T) L + c_2(T) + o(1)$ □.

— Pfaffian Case this argument starts to fail for us!??

(4) Kac Trick $\quad T \geq 0$ and $\int T(x)\,dx = 1$.

T-I Case $\quad (S_N)$ simple random walk. P_x – start at x.

T symmetric $\quad (S_N - S_{N-1})$ IID $\sim T(x)\,dx$.



$$\frac{d}{dL} \mathrm{Tr}(K^M) = M \int_{[0,L]^{M-1}} T(x_2 - L)\, T(x_3 - x_2) \dots T(x_M - x_{M-1})\, T(L - x_M)\, dx_2 \dots dx_M$$

$$= M\, P_L[\, S_k \in [0,L]\ k = 1, \dots, N,\ \delta_L(S_M)\,]$$

$$= M\, P_L(\tau = M,\ \delta_L(S_\tau)) \qquad \tau = \tau_{[0,L]^c} \text{ exit from } [0,L]$$

So $\displaystyle \frac{d}{dL} \sum_{M=1}^{\infty} \frac{p^M}{M} \mathrm{Tr}(K^M) = \mathbb{E}_L[\, p^\tau \delta_L(S_\tau)\,].$ Wow!

Probabilistic Exact.

Similarly ...

$$\ln \mathrm{Det}(I - pK) = -\mathbb{E}_0\left[\, p^\tau \delta_0(S_\tau)\, (L - \max_{k \leq \tau} S_k)_+ \right]$$ for $p \in [0,1]$ small

Extend to all p. $\mathrm{Det}(I - pK)$ analytic in $p \in \mathbb{R}$.
RHS Also converges for $p \in [0,1]$.

The

$(L-M)_+ = L - \min(L, M)$, $M_N = \max_{k \le N} S_k$, $\tau = \cancel{\inf}\ T_{[0,L]^c}$

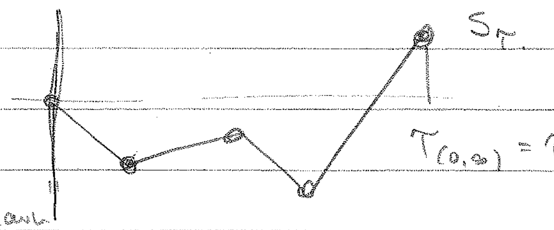
$$\log \operatorname{Det}_{[0,L]}(1 - pK) = L\,\mathbb{E}_0[p^\tau \delta_0(S_\tau)] - \mathbb{E}[p^\tau \delta_0(S_\tau)\min(L, M_\tau)]$$

$$= L\,\underbrace{\mathbb{E}_0[p^\tau \delta_0(S_\tau)]}_{c_1} - \underbrace{\mathbb{E}[p^\tau \delta_0(S_\tau) M_\tau]}_{c_2} + o(1).$$

Except if $p = 1$ the $\mathbb{E}[\delta_0(S_\tau)M_\tau] = \infty$

Correct $Lc_1 - \log L + c_2 + o(1)$

(5) Exact formulae from S.R.W



FACT Under P_0, law of (τ, S_τ) "known"

under P

$$\mathbb{E}_0[p^\tau e^{ikS_\tau}] = 1 - \exp\left\{-\sum_{N=1}^{\infty} \frac{p^N}{N} \int_0^\infty e^{ikx} f^{*n}(x)\,dx\right\}$$

Feller (f non symmetric)

Under P_x get integral eqn by conditions on one step over $(-\infty, 0)$.

Wiener–Hopf for speed τ

e.g. $\mathbb{E}_0[p^\tau] = 1 - \sqrt{1-p}$! $\qquad \mathbb{E}_0[S_\tau] = \frac{\mathbb{E}(S_1^2)}{\sqrt{2}}$

SPARRE ANDERSON ~~FORMULA~~

f symmetric

$\longrightarrow$ .. Explicit formulae for c_1, c_2 in terms of $\boxed{f^{*n}}$.

$\boxed{\text{Now remove } f \ge 0}$

(6) Pfaffian Case. $\quad \mathbb{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} S(x,y) + K(x,y) & -D_2K(x,y) \\ -D_1K(x,y) & D_{12}K(x,y) \end{pmatrix} \quad x < y$

~~$(K_{ij}(x,y) = -K_{ji}(y,x))$~~

Derivative form. GOE / ~~Sym~~ Real G. / ARM...
↓
GAUSSIAN

~~$S(x,y) = \operatorname{sgn}(y-x)$.~~

TRACY WIDOM MANIPULATION

→ G.O.E.

$$\left(\mathrm{Pf}(J - pK)\right)^2 = \mathrm{Det}(J - pK) = \mathrm{Det}(I + pJK)$$

EXECUTIVE SUMMARY

$$JK = \begin{pmatrix} 0 & \partial \\ -I & I \end{pmatrix}\begin{pmatrix} S & 0 \\ -K & D_2K \end{pmatrix} \xrightarrow{\text{SYLVESTER'S}} \begin{pmatrix} S & 0 \\ -K & D_2K \end{pmatrix}\begin{pmatrix} 0 & \partial \\ -I & I \end{pmatrix}$$

$$\Downarrow$$

$$\mathrm{Det}(I + pJK) \xleftarrow{\text{on } (L^2[a,b])^2} \begin{pmatrix} 0 & S\partial \\ -D_2K & -K\partial + D_2K \end{pmatrix}$$

BLOCK MATRIX

$$= \mathrm{Det}(1 - pK\partial + pD_2K + p^2 D_2KS\partial) \quad \text{on } L^2[a,b]$$

$$= \mathrm{Det}(1 + 2p(1-p)D_2K + F) \qquad \underline{F = \text{Rank } 2}\,!$$

$$= \mathrm{Det}(1 + 2p(1-p)D_2K)\ \det\left(1 + (1 + 2p(1-p)D_2K)^{-1}F\right)$$

2x2 matrix determinant

- $\underline{\text{T.I. CASES}}$ $\quad -\tfrac{1}{2}D_2K = f(y-x)$ $\quad \underline{f \geq 0 \ \int f = 1, \text{ some tail decay.}}$

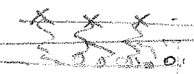
PROB FORMULAE FOR Det, det in terms of S_N

Asymptotics for $p \in (0, \tfrac{1}{2})$, $p = \tfrac{1}{2}$, $p \in (\tfrac{1}{2}, 1)$ different arguments!

— $\underline{\text{No logs.}}$ — $\underline{\text{Leading order comes from both terms } p \in (\tfrac{1}{2}, 1)}$

- $\underline{\text{NOT T-I CASES}}$ for point processes at $\underline{\text{edge}}$ eg. GR Ginibre Edge

eg. ABM from half space coalescing



$$-\tfrac{1}{2}D_2K(x,y) \Rightarrow = \int_{-\infty}^{0} f(x-z)f(y-z)\,dz \qquad f \text{ prob density}$$

$\underline{\text{Prob interpretation again.}}$ Are formulae true without $f \geq 0$?

- $\log \mathrm{Pf}_{[0,L]}(J - pK) = -c_1(p)L + c_2(p) + o(1)$ as $L \to \infty$

- $\log \mathrm{Pf}_{[-L,\infty)}(J - pK)$ as $L \to \infty$