# Example

## 1 Gaussian integrals

Recall: let $A$ be a $2n \times 2n$ matrix s.t. $A^T = -A$. Let

$$\omega = \sum_{i<j} a_{ij}\, \theta_i \theta_j \in \Lambda(\mathbb{R}^{2n})$$

Then $\frac{1}{n!}\omega^n =: Pf(A)\, \theta_1 \dots \theta_n$

Therefore

$$\int_{\mathbb{R}^{0|2n}} d\theta\, e^{\frac{1}{2}\sum_{i,j} a_{ij}\theta_i\theta_j} = \int_{\mathbb{R}^{0|2n}} d\theta \sum_{k=0}^{\infty} \frac{1}{k!}\omega^k$$

$$= \int_{\mathbb{R}^{0|2n}} d\theta\, \frac{1}{n!}\omega^n = Pf(A).$$

Integrals over $\mathbb{R}^{m|n}$

$$\int_{\mathbb{R}^{m|n}} dx\, d\theta\, f(x,\theta) := \int_{\mathbb{R}^{m|0}} dx \left( \int_{\mathbb{R}^{0|n}} d\theta\, f(x,\theta) \right)$$

Changes of variables are expressed in terms of superdeterminants.

## Example

## 2 Localization.

$$\int_{\mathbb{R}^{2|2}} dx\, d\theta\; f\left(x_1^2 + x_2^2 + \theta_1 \theta_2\right)$$

$$= \int_{\mathbb{R}^{2|0}} dx \int_{\mathbb{R}^{0|2}} d\theta_2\, d\theta_1\; f'\left(x_1^2 + x_2^2\right) \theta_1 \theta_2$$

$$= \int_{\mathbb{R}^{2|0}} dx\; f'\left(x_1^2 + x_2^2\right) = \int_0^{2\pi} d\varphi \int_0^{\infty} dr\; r\, f'(r^2)$$

$$= \pi \int_0^{\infty} dx\; f'(x) = -\pi f(0)$$

(The integral is determined by $f(0)$ provided it converges)

More generally, let $O(2|2)$ be the group of linear transformations of $\mathbb{R}^{(2|2)}$ which preserves $x_1^2 + x_2^2 + \theta_1 \theta_2$.

Then

$$\int_{\mathbb{R}^{n|2}} dx\, d\theta\; \underbrace{f(x, \theta)}_{\substack{\text{Invariant} \\ \text{wrt } O(2,2) \\ \text{acting on } \mathbb{R}^{2|2} \subset \mathbb{R}^{n|2}}} = -\pi \int_{\mathbb{R}^{n-2}} dx\; f(x, \theta) \Big|_{\substack{x_1 = x_2 = 0 \\ \theta_1 = \theta_2 = 0}}$$

Example 3 Determinant as a Gaussian Berezin integral.

S'pose $A = \left(\begin{array}{c|c} 0 & M \\ \hline -M^T & 0 \end{array}\right)$

Then $\displaystyle\int_{\mathbb{R}^{0|2n}} d\theta\, e^{\frac{1}{2}\sum_{ij} a_{ij}\theta_i\theta_j} = Pf(A) = (-1)^{\frac{n(n-1)}{2}} \det M$

$$\int \underbrace{d\theta_{2n}\, d\theta_{2n-1} \cdots d\theta_{n+1}}_{d\bar{\psi}_n \ldots d\bar{\psi}_1} \underbrace{d\theta_n \ldots d\theta_1}_{d\psi_n \ldots d\psi_1} e^{\frac{1}{2}\sum_{ij} a_{ij}\theta_i\theta_j} = (-1)^{\frac{n(n-1)}{2}} \det M$$

# $\displaystyle\sum_{ij} a_{ij}\theta_i\theta_j = (\psi^T\ \bar{\psi}^T)\left(\begin{array}{c|c} 0 & M \\ \hline -M & 0 \end{array}\right)\begin{pmatrix}\psi \\ \bar{\psi}\end{pmatrix}$

$$= \left(-\bar{\psi}^T M \quad \psi^T M\right)\begin{pmatrix}\psi \\ \bar{\psi}\end{pmatrix} = -2\bar{\psi}^T M \psi$$

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So $\displaystyle\int d\bar{\psi}\, d\psi\, e^{-\langle \bar{\psi}, M\psi\rangle} = (-1)^{\frac{n(n-1)}{2}} \det M$