Thm 2.8 (Bakry–Émery) Let ν:
$$\mathbb{E}_\nu[F(\varphi)] = \frac{1}{Z}\int_{\mathbb{R}^N} e^{-H(\varphi)} F(\varphi)\, d\varphi$$
be the prob. measure on $X = \mathbb{R}^N$, where $\mathrm{Hess}\, H(\varphi) \geq \lambda \cdot I \quad \forall \varphi \in X$. Then ν satisfies the $LSI(\rho)$ with $\rho \geq \lambda$.

Proof. ν can be characterized as the invariant measure for a Markov process with the generator
$$\Delta^H = \Delta - (\nabla H, \nabla), \quad \text{where}$$
$$\Delta = \sum_{x\in\Lambda} \frac{\partial^2}{\partial \varphi_x^2}, \quad (\nabla H, \nabla) = \sum_{x\in\Lambda} \frac{\partial H}{\partial \varphi_x}\frac{\partial}{\partial \varphi_x}$$

$|\Lambda| = N$

The corresponding semigroup P_t is defined by
$$P_t F(\varphi) := \mathbb{E}_{\varphi_0 = \varphi}[F(\varphi_t)]$$

If $F_t := P_t F$, then $\frac{\partial F_t}{\partial t} = \Delta^H F_t$
(backward Kolmogorov).

If m_t is the distribution at time t,
$$\mathbb{E}_{m_t}[F_0] = \mathbb{E}_{m_0}[F_t] \qquad F_0 \equiv F$$
$m_t \xrightarrow{t\uparrow\infty} \nu$.

Then
$$Ent_\nu(F) := \mathbb{E}_\nu[\phi(F)] - \phi(\mathbb{E}_\nu[F])$$

$\phi(x) = x\log x$.

$$= \mathbb{E}_\nu[\phi(F_0)] - \phi(\mathbb{E}_\nu[F_0])$$

Ergodicity: $\forall F_0 \in L^2(\nu)$, $F_t \to \mathbb{E}_\nu[F_0]$ in $L^2(\nu) \Rightarrow$ For any bounded smooth $F_0 : X \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $\lim_{t\to\infty} \mathbb{E}_\nu[g(F_t)] = g(\mathbb{E}_\nu[F_0])$

$$\underset{\substack{\nu \text{ is inv,} \\ \text{Ergodicity}}}{=} \mathbb{E}_\nu[\phi(F_0)] - \mathbb{E}_\nu[\phi(F_\infty)]$$

(Some analysis needed, as ϕ isn't bounded on $(0,\infty)$)

Therefore,

$$\mathrm{Ent}_\nu(F) = -\int_0^\infty dt \, \frac{\partial}{\partial t} \mathbb{E}_\nu\left[\Phi(F_t)\right]$$

$$\overset{\text{Prop. 2.2}}{=} \int_0^\infty dt \; I_\nu(F_t),$$

where $I_\nu(F_t) := \mathbb{E}_\nu\left[\frac{(\nabla F_t)^2}{F_t}\right]$

is the Fisher information.

Claim. $I_\nu(F_t) \leq e^{-2\lambda t} I_\nu(F_0)$

The claim implies that

$$\mathrm{Ent}_\nu(F) \leq \int_0^\infty dt \; e^{-2\lambda t} I_\nu(F_0) = \frac{1}{2\lambda} I_\nu(F_0)$$

$$= \frac{2}{\lambda} D_\nu(\sqrt{F}) \Rightarrow \text{log-Sobolev with } \rho = \lambda.$$

Proof of the claim: use Gronwell.

$$\frac{\partial}{\partial t} I_\nu(F_t) = \frac{\partial}{\partial t} \mathbb{E}_\nu\left[\frac{(\nabla F_t)^2}{F_t}\right] \overset{\text{and.}}{=} \mathbb{E}_\nu\left[\frac{\partial}{\partial t} \frac{(\nabla F_t)^2}{F_t}\right]$$

ν is invariant, so $\mathbb{E}_\nu[\Delta_H G] = 0$, provided G is "nice"

$$= \mathbb{E}_\nu\left[\left(\frac{\partial}{\partial t} - \Delta^H\right) \frac{(\nabla F_t)^2}{F_t}\right]$$

Notice: $\frac{\partial}{\partial t} F_t = \Delta^H F_t$

$$\# \quad \frac{\partial}{\partial t} \frac{(\nabla F)^2}{F} = -\frac{1}{F^2}(\nabla F)^2 \Delta^H F + 2\frac{(\nabla F, \nabla \Delta^H F)}{F}$$

$$= -\frac{1}{F^2}(\nabla F)^2\big(\Delta F - (\nabla H, \nabla F)\big) + 2\frac{(\nabla F, \nabla \Delta F)}{F} - 2\frac{\overset{Ⓐ}{(\nabla F, \nabla(\nabla H, \nabla F))}}{F}$$

$$Ⓐ = \sum_{ij} \partial_i F\, \partial_i(\partial_j H \partial_j F) = (\nabla F, \mathrm{Hess}\, H\, \nabla F) + (\nabla F, \mathrm{Hess}\, F\, \nabla H)$$

$$\frac{\partial}{\partial t}\frac{(\nabla F)^2}{F} = -\frac{1}{F^2}(\nabla F)^2\big(\Delta F - (\nabla H, \nabla F)\big) + 2\left(\frac{\nabla F, \nabla \Delta F}{F}\right)$$

$$-2\frac{1}{F}\big((\nabla F, \mathrm{Hess}\, H\, \nabla F) + (\nabla F, \mathrm{Hess}\, F\, \nabla H)\big)$$

$$\Delta^H \frac{(\nabla F)^2}{F} = \Delta \frac{(\nabla F)^2}{F} - (\nabla H, \nabla)\frac{(\nabla F)^2}{F}$$

$$= \Delta \frac{(\nabla F)^2}{F} + \frac{1}{F^2}(\nabla F)^2(\nabla H, \nabla F) - 2\frac{(\nabla F, \mathrm{Hess}\, F\, \nabla H)}{F}$$

$$\left(\frac{\partial}{\partial t} - \Delta^H\right)\frac{(\nabla F)^2}{F} = -\frac{1}{F^2}(\nabla F)^2 \Delta F + 2\frac{(\nabla F, \nabla \Delta F)}{F}$$

$$-2\frac{1}{F}(\nabla F, \mathrm{Hess}\, H\, \nabla F) - \Delta \frac{(\nabla F)^2}{F}$$

$$= -2F(\nabla \log F, \mathrm{Hess}\, H\, \nabla \log F) + R$$

$$R = -\frac{1}{F^2}(\nabla F)^2 \Delta F + \frac{2}{F}(\nabla F, \nabla \Delta F) - \underbrace{\Delta \frac{(\nabla F)^2}{F}}_{X}$$

$$X = \sum_i \partial_i\left(\frac{2(\nabla F, \nabla \partial_i F)}{F} - \frac{(\nabla F)^2}{F^2}\partial_i F\right)$$

$$= \frac{2}{F}(\nabla F, \nabla \Delta F) + \frac{2}{F}|\mathrm{Hess}\, F|_2^2 - \frac{4}{F^2}\sum_i (\nabla F, \nabla \partial_i F)\partial_i F$$

$$-\frac{(\nabla F)^2}{F^2}\Delta F - \frac{2(\nabla F, \nabla \partial_i F)\partial_i F}{F^2} + 2\frac{(\nabla F)^2}{F^3}(\nabla F, \nabla F)$$

$$R = -\frac{2}{F}\,|\mathrm{Hess}\,F|_2^2 + \frac{4}{F^2}\left(\nabla F,\, \mathrm{Hess}\,F\,\nabla F\right) - 2\,\frac{(\nabla F, \nabla F)^2}{F^3}$$

On the other hand,

$$2F\,|\mathrm{Hess}\log F|_2^2 = 2F\sum_{ij}\left(\partial_i\partial_j \log F\right)^2$$

$$= 2F\sum_{ij}\left(\partial_i \frac{\partial_j F}{F}\right)^2 = 2F\sum_{ij}\left(\frac{1}{F}\partial_i\partial_j F - \frac{1}{F^2}\partial_i F\partial_j F\right)^2$$

$$= 2F\sum_{ij} \frac{1}{F^2}(\partial_i\partial_j F)(\partial_i\partial_j F)$$

$$- 2\,\frac{1}{F^3}(\partial_i\partial_j F)\,\partial_i F\partial_j F + \frac{1}{F^4}\partial_i F\partial_j F\partial_i F\partial_j F$$

$$= \frac{2}{F}\,|\mathrm{Hess}\,F|_2^2 - \frac{4}{F^2}\left(\nabla F,\, \mathrm{Hess}\,F\,\nabla F\right) + \frac{2}{F^3}\left(\nabla F, \nabla F\right)^2$$

$$= R$$

Conclusion:

$$\left(\frac{\partial}{\partial t} - \Delta^H\right)\frac{(\nabla F)^2}{F} = -2F\underbrace{|\mathrm{Hess}\log F|_2^2}_{\geq 0} - 2F\left(\nabla \log F,\, \underbrace{\mathrm{Hess}\,H}_{\geq \lambda \mathrm{Id}}\nabla \log F\right)$$

$$\leq -2\lambda\, F\left(\nabla \log F\right)^2 = -2\lambda\,\frac{(\nabla F)^2}{F}$$

#

Conclusion:

$$\left(\frac{\partial}{\partial t}\right) I_\nu(F_t) \leq -2\lambda\, \mathbb{E}_\nu\left[\frac{(\nabla F_t)^2}{F_t}\right] = -2\lambda I_\nu(F_t)$$

$$\Rightarrow \quad I_\nu(F_t) \leq e^{-2\lambda t} I_\nu(F_0)$$

The claim is proved

The theorem is proved.

Intro to Polchinski equation.

1. Gaussian integration.

Let $C \in \mathbb{R}^{N\times N}$ be positive semi-definite, let $\mathbb{P}_C$ be the corresp. centred Gaussian measure with covariance C, $\mathrm{supp}(\mathbb{P}_C) = \mathrm{Im}(C)$

(1) $$\mathbb{E}_C[F] \propto \int_{\mathbb{R}^N} e^{-\frac{1}{2}(\xi, C^{-1}\xi)} F(\xi)\, d\xi$$

$\mathbb{R}^N \leftarrow$ If C is PD Otherwise replace with $\mathrm{Im}(C)$

Remark: it is useful to work with semi-definite covariance matrices. Then one can consider (potentially) large-N limits of random matrices.

Semigroup property. If $C = C_1 + C_2$, (Positive s-definite)

(2) $$\mathbb{E}_C[F(\xi)] = \mathbb{E}_{C_2}\left[\mathbb{E}_{C_1}[F(\xi_1 + \xi_2)]\right],$$

where $\xi_i \sim N(0, C_i)$ Independent

\# It is enough to check (2) for

$$\xi \mapsto F(\xi) = e^{(\lambda, \xi)}$$ for any $\lambda \in \mathbb{R}^N$

Then $\mathbb{E}_{C_2}\left[\mathbb{E}_{C_1}[F(\xi_1 + \xi_2)]\right]$

$$= \mathbb{E}_{C_2}\left[\mathbb{E}_{C_1}\left[e^{(\lambda, \xi_1 + \xi_2)}\right]\right] = \mathbb{E}_{C_2}\left[e^{(\lambda, \xi_2)} \mathbb{E}_{C_1}\left[e^{(\lambda, \xi_1)}\right]\right]$$

$$= e^{\frac{1}{2}(\lambda, C_2 \lambda)} e^{\frac{1}{2}(\lambda, C_1 \lambda)} = e^{\frac{1}{2}(\lambda, (C_1 + C_2)\lambda)}$$

$$= e^{\frac{1}{2}(\lambda, C\lambda)} = E_C\left[e^{(\lambda,\xi)}\right] \qquad \#$$

Notations. $\bullet\ \Delta_C := \sum_{i,j} C_{ij} \frac{\partial^2}{\partial y_i \partial y_j}$; (3)

(4) $\bullet\ (\ ,\ )_C : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$

$$(u, v) \longmapsto (u,v)_C = \sum_{i,j} C_{ij} u_i v_j$$

$\bullet\ |u|_C^2 = (u)_C^2 = (u,u)_C.$

$\bullet\ (u,v) = \sum_i u_i v_i$

Any closed interval can be used

Principal object: $t \in [0, +\infty] \longmapsto C_t$ - continuous increasing function into positive semi-definite matrices on $\mathbb{R}^N$ (increasing in the sense of quadratic forms). More precizely,

$$C_t = \int_0^t \dot{C}_s \, ds,$$

where $s \mapsto \dot{C}_s$ is cadlag taking values in positive semidefinite matrices.

Say that $C_\infty = \int_0^\infty \dot{C}_s \, ds$ is a covariance decomposition of C_∞.

Let $X \subset \mathbb{R}^N$ be the image of C_∞.

(Useful remark: (image C_{t_1}) $\subset$ (image C_{t_2}) $\subset X$ for $t_1 < t_2$)

Let X_t be the image of C_t, $t \geq 0$

lec 1

Proposition 3.1. For $F \in C^2(X, \mathbb{R})$, define

$$F_t = \mathbb{P}_{C_t} * F, \text{ i.e.}$$

$$F_t(\varphi) := \mathbb{E}_{C_t}\left[F(\varphi+\xi)\right]$$

Then for all t which are not disc points of $\dot{C}_t$,

(5) $$\frac{\partial}{\partial t} F_t = \frac{1}{2} \Delta_{\dot{C}_t} F_t, \quad F_0 = F$$

(6) Or $\dot{P}_{C_t} = \frac{1}{2} \Delta_{\dot{C}_t} P_{C_t}$ (as a measure)

\#

$$F_0(\varphi) = \mathbb{E}_{\underset{0}{\overset{\shortparallel}{C_0}}}\left[F(\varphi+\xi)\right] \underset{\xi=0 \text{ a.s.}}{=} F(\varphi) \text{ O.K.}$$

If t is not a disc point, $\exists$ ε s.t. X_s is indep. of $s \in [t-\varepsilon, t+\varepsilon]$, C_s is strictly positive definite for any $s \in [t-\varepsilon, t+\varepsilon]$. Then (1) is true on X_s for any such s and (5) follows from the defn of $\mathbb{E}_C$

Alternatively, can use Ito's calculus:

Define

$$\xi_t = \varphi + \int_0^t \sqrt{\dot{C}_s}\, dB_s \in \mathbb{R}^N, \quad t \geq 0$$

Then $F_t(\varphi) = \mathbb{E}\left[F(\xi_t)\right]$ and

$$d\,F(\xi_t) = \nabla F(\xi_t)\, d\xi_t + \frac{1}{2} \nabla \otimes \nabla F(\xi_t)\, d\xi_t \otimes d\xi_t$$

$$= \nabla F(\xi_t)\, d\xi_t + \frac{1}{2} \Delta_{\dot{C}_t} F(\xi_t) \Rightarrow$$

φ-derivatives

$$\begin{cases} \dot{F}_t(\varphi) = \frac{1}{2} \Delta_{\dot{C}_t} F_t(\varphi) \\ F_0(\varphi) = \mathbb{E}\left[F(\xi_0)\right] = F(\varphi) \end{cases} \quad \#$$

2. Renormalized potential and the Polchinski equation.

Let ν_0 be a prob. measure on $X \subset \mathbb{R}^N$ linear

Let $C_\infty = \int_0^\infty \dot{C}_t \, dt$ be a covariance decomp.

Suppose that the ν_0 expectation can be presented in the form of a Gaussian expectation:

(10) $$\mathbb{E}_{\nu_0}[F] = \frac{1}{Z} \mathbb{E}_{C_\infty}\left[e^{-V_0(\zeta)} F(\zeta)\right],$$

where $V_0 : X \to \mathbb{R}$ is a potential assumed to be bounded below

$$\left(Z = \mathbb{E}_{C_\infty}\left[e^{-V_0(\zeta)}\right] \right)$$

Definition 2: For $t > s > 0$ and bounded $F : X \to \mathbb{R}$ and $\varphi \in X$, define

- the renormalized potential V_t:

(11) $$\boxed{V_t(\varphi) = -\log \mathbb{E}_{C_t}\left[e^{-V_0(\varphi+\zeta)}\right]}$$

- the Polchinski semigroup $P_{s,t}$:

(12) $$\boxed{P_{s,t} F[\varphi] = e^{V_t(\varphi)} \mathbb{E}_{C_t - C_s}\left[e^{-V_s(\varphi+\zeta)} F(\varphi+\zeta)\right]}$$

- the renormalized measure ν_t:

(13) $$\boxed{\begin{aligned}\mathbb{E}_{\nu_t}[F] &= P_{t,\infty} F(0) \\ &= e^{V_\infty(0)} \mathbb{E}_{C_\infty - C_t}\left[e^{-V_t(\zeta)} F(\zeta)\right]\end{aligned}}$$

Remark. Consistency of (13):

$$1 = \mathbb{E}_{\nu_t}[1] = e^{V_\infty(0)}\, \mathbb{E}_{C_\infty - C_t}\left[e^{-V_t(\xi)}\right]$$

$$\underset{\text{Defn of ren. pot}}{=} e^{V_\infty(0)}\, \mathbb{E}_{C_\infty - C_t}\left[\mathbb{E}_{C_t}\left[e^{-V_0(\xi_1+\xi_2)}\right]\right]$$

$$\underset{\xi_1 \perp \xi_2}{=} e^{V_\infty(0)}\, \underbrace{\mathbb{E}_{C_\infty}\left[e^{-V_\infty(\xi)}\right]}_{e^{-V_\infty(0)}} = 1 \quad \text{O.K.}$$

Remark. The renormalized potential and moment generating function:

$$V_\infty(C_\infty h) \overset{(11)}{=} -\log \mathbb{E}_{C_\infty}\left[e^{-V_0(C_\infty h+\xi)}\right]$$

Gaussian change of variables $\xi \to \xi - C_\infty h$

$$\overset{\text{Gaussian}}{=} -\log \mathbb{E}_{C_\infty}\left[e^{-V_0(\xi)}\, e^{(h,\xi)}\, e^{-\frac{1}{2}(h, C_\infty h)}\right]$$

$$= \frac{1}{2}(h, C_\infty h) - \log \mathbb{E}_{C_\infty}\left[e^{-V_0(\xi)}\, e^{(h,\xi)}\right]$$

$$= \frac{1}{2}(h, C_\infty h) - \log \underbrace{\mathbb{E}_{C_\infty}\left[e^{-V_0(\xi)}\right]}_{e^{-V_\infty(0)}} \mathbb{E}_{\nu_0}\left[e^{(h,\xi)}\right]$$

$$= V_\infty(0) + \frac{1}{2}(h, C_\infty h) - \log \mathbb{E}_{\nu_0}\left[e^{(h,\xi)}\right]$$

Proposition 3. For $t \geq 0$ and any $F : X \to \mathbb{R}$ s.t. the following quantities make sense,

$$\mathbb{E}_{\nu_0}[F] = \mathbb{E}_{\nu_t}[P_{0,t} F]$$

Proof.

$$\mathbb{E}_{V_0}[F] = \frac{1}{\mathbb{E}_{C_\infty}[e^{-V_0(\xi)}]} \mathbb{E}_{C_\infty}\left[e^{-V_0(\xi)} F(\xi)\right]$$

$$= \frac{1}{\mathbb{E}_{C_\infty}[e^{-V_0(\xi)}]} \mathbb{E}_{C_\infty - C_t}\left[\mathbb{E}_{C_t}\left[e^{-V_0(\varphi+\xi)} F(\xi+\varphi)\right]\right]$$

$$= \frac{1}{\mathbb{E}_{C_\infty}[e^{-V_0(\xi)}]} \mathbb{E}_{C_\infty - C_t}\left[e^{-V_t(\varphi)} P_{0,t} F(\varphi)\right] =$$

$P_{0,t} F(\varphi) = e^{V_t(\varphi)} \mathbb{E}_{C_t}\left[e^{-V_0(\varphi+\xi)} F(\xi+\varphi)\right]$

$$= \frac{e^{-V_\infty(0)}}{\mathbb{E}_{C_\infty}[e^{-V_0(\xi)}]} \mathbb{E}_{V_t}[F]$$

= 1 (defn of the RP) QED

Remarks:

1. (3.12) $\Rightarrow$ $P_{0,t} F(\varphi) = e^{V_t(\varphi)} \mathbb{E}_{C_t}\left[e^{-V_0(\varphi+\xi)} F(\varphi+\xi)\right]$

$$\overset{(3.11)}{=} \frac{\mathbb{E}_{C_t}\left[e^{-V_0(\varphi+\xi)} F(\varphi+\xi)\right]}{\mathbb{E}_{C_t}\left[e^{-V_0(\varphi+\xi)}\right]} := \mathbb{E}_{\mu_t^{\varphi}}\left[F(\xi)\right]$$

Cond. expectation.

μ_t^{φ} is called the fluctuation measure.

Equivalently,

$$\# \quad \frac{1}{Z}\int d\xi \; e^{-\frac{1}{2}\langle \xi, C_t^{-1}\xi\rangle} e^{-V_0(\varphi+\xi)} F(\varphi+\xi)$$

$$= \frac{1}{Z}\int d\xi \; e^{-\frac{1}{2}\langle (\xi-\varphi), C_t^{-1}(\xi-\varphi)\rangle} e^{-V_0(\xi)} F(\xi)$$

$$= \frac{1}{Z}\int d\xi \; e^{-\frac{1}{2}\langle \xi, C_t^{-1}\xi\rangle - \frac{1}{2}\langle \varphi, C_t^{-1}\varphi\rangle} e^{\langle \xi, C_t^{-1}\varphi\rangle} e^{-V_0(\xi)} F(\xi)$$

$$\mu_t^{\varphi}(d\xi) \sim e^{V_t(\varphi) - \frac{1}{2}\langle \varphi, C_t^{-1}\varphi\rangle} \quad \# \quad e^{-\frac{1}{2}\langle \xi, C_t^{-1}\xi\rangle + \langle \xi, C_t^{-1}\varphi\rangle - V_0(\xi)} d\xi$$

- similar to the original measure, but with $C_\infty \to C_t < C_\infty$ $\Rightarrow$ "More convex measure"

2. Proposition 3.3 $\Rightarrow$

$$\mathbb{E}_{\nu_0}[F] = \mathbb{E}_{\nu_t}\left[P_{0,t} F\right] \overset{①}{=} \mathbb{E}_{\nu_t}\left[\mathbb{E}_{\mu_t^{\varphi}}\left[F(\xi)\right]\right]$$

Over φ — Cov $C_\infty - C_t$ "slow scales"

Over ξ — Cov. C_t "fast scales"

- an instance of measure decomposition, see section 2.5

Multiscale renormalization $\to$
infinitesimal renormalization steps $\to$
HJ equation for V_t, known as
Polchinski equation

Prop. 3.4. Let $(C_t)_{t\geq 0}$ be a covariance decomposition, let $V_0 \in C^2$. Then $\forall t$: C_t is diff, the renorm. potential V_t satisfies the Polchinsky equation

$$\frac{\partial}{\partial t} V_t = \frac{1}{2} \Delta_{\dot{C}_t} V_t - \frac{1}{2} (\nabla V_t)^2_{\dot{C}_t}$$

Proof. Define $Z_t(\varphi) = \mathbb{E}_{C_t}\left[e^{-V_0(\zeta+\varphi)}\right]$

$$\text{Prop 3.1} \Rightarrow \begin{cases} \frac{\partial}{\partial t} Z_t = \frac{1}{2} \Delta_{\dot{C}} Z_t \\ Z_0 = e^{-V_0} \end{cases}$$

$Z_0 > 0 \Rightarrow Z_t > 0 \Rightarrow V_t = -\log Z_t$ is well defined and

$$\dot{V}_t = -\frac{1}{Z_t} \dot{Z}_t = -\frac{1}{Z_t} \Delta_{\dot{C}_t} Z_t$$

$$= -\frac{1}{2} e^{V_t} \Delta_{\dot{C}_t} e^{-V_t}$$

$$= -\frac{1}{2} e^{V_t} \sum_{ij} (\dot{C}_t)_{ij} \partial_i \partial_j e^{-V_t}$$

$$= \frac{1}{2} \Delta_{\dot{C}} V_t - \frac{1}{2} (\nabla V_t, \nabla V_t)_{\dot{C}_t}$$

#

Proposition 3.5 Let V_0 be bounded below. The operators $(P_{s,t})_{s\le t}$ form a time-dependent markov semigroup with generators (L_t) defined below in the sense that

$$P_{t,t} = Id$$
$$P_{r,t} P_{s,r} = P_{s,t}, \quad s \le r \le t$$

and $\begin{cases} P_{s,t} F \ge 0 \text{ if } F \ge 0, \quad s \le t \\ P_{s,t} 1 = 1 \end{cases}$

at all points t at which C_t is diff. (C_s is differentiable),

$$\frac{\partial}{\partial t} P_{s,t} F = L_t P_{s,t} F,$$
$$-\frac{\partial}{\partial s} P_{s,t} F = P_{s,t} L_s F \quad (s \le t)$$

where $$L_t F = \frac{1}{2} \Delta_{\dot{C}_t} F - (\nabla V_t, \nabla F)_{\dot{C}_t}$$

The evolution of V_t is dual to $(P_{s,t})$:

$$\begin{cases} \mathbb{E}_{V_t}[P_{s,t} F] = \mathbb{E}_{V_s}[F] \quad (s \le t) \\ -\frac{\partial}{\partial t} \mathbb{E}_{V_t}[F] = \mathbb{E}_{V_t}[L_t F] \end{cases}$$

Proof Recall the definition:

$$(*) \quad P_{s,t} F(\varphi) = e^{V_t(\varphi)} \mathbb{E}_{C_t - C_s}\left[e^{-V_s(\varphi+\zeta)} F(\varphi+\zeta)\right]$$

$$\Rightarrow P_{t,t} F(\varphi) = \lim_{s \uparrow t} \underbrace{e^{V_t(\varphi)}}_{\text{cont}} \underbrace{\mathbb{E}_{C_t - C_s}}_{\substack{\text{Goes to} \\ \delta_0 \text{ measure}}}\left[e^{-V_s(\varphi+\zeta)} F(\varphi+\zeta)\right] = e^{V_0(\varphi)} e^{-V_0(\varphi)} F(\varphi)$$
$$= F(\varphi)$$

Semigroup – generalize the splitting argument used before:

$$P_{s,t}F(\varphi) = e^{V_t(\varphi)}\,\mathbb{E}_{C_t - C_s}\left[e^{-V_s(\varphi+\xi)}F(\varphi+\xi)\right]$$

$$= e^{V_t(\varphi)}\,\mathbb{E}_{C_t - C_r + C_r - C_s}\left[e^{-V_s(\varphi+\xi)}F(\varphi+\xi)\right]$$

$$= e^{V_t(\varphi)}\,\mathbb{E}_{\underset{\sim \xi_2}{C_t - C_r}}\left[\mathbb{E}_{\underset{\sim \xi_1}{\underbrace{C_r - C_s}}}\left[e^{-V_s(\varphi+\xi_2+\xi_1)}F(\varphi+\xi_2+\xi_1)\right]\right]$$

$$= e^{V_t(\varphi)}\,\mathbb{E}_{C_t - C_r}\, e^{-V_r(\varphi+\xi_2)}\,\mathbb{P}_{s,r}F(\varphi+\xi_2)$$

$$= \mathbb{P}_{r,t}\,\mathbb{P}_{s,r}\,F(\varphi)$$

Positivity follows from the defn. $(*)$

Continuity: also from the definition, as

$$\|\mathbb{P}_{s,t}F\|_\infty = \left\|e^{V_t(\varphi)}\,\mathbb{E}_{C_t - C_s}\left[e^{-V_s(\varphi)}F(\varphi+\xi)\right]\right\|_\infty \leq \left\|e^{V_t(\varphi)}\,\mathbb{E}_{C_t - C_s}\left[e^{-V_s(\varphi)}\right]\right\|_\infty$$

$$\|F\|_\infty = \|F\|_\infty \quad (\text{Conclusion ?})$$

$$\frac{\partial}{\partial t}P_{s,t}F(\varphi) = \partial_t V_t(\varphi)\,P_{s,t}F(\varphi)$$

$$+\, e^{V_t}\,\Delta_{\dot{C}_t}\,\mathbb{E}_{C_t - C_s}\left[e^{-V_s(\xi+\varphi)}F(\varphi+\xi)\right]$$

$$= \partial_t V_t(\varphi)\,P_{s,t}F(\varphi) + e^{V_t}\tfrac{1}{2}\Delta_{\dot{C}_t}\,e^{-V_t}\left(P_{s,t}F(\varphi)\right)$$

$$\overset{PE}{=} \left(\frac{1}{2}\Delta_{\dot{c}_t}\cancel{V_t} - \frac{1}{2}(\nabla V_t)^2_{\dot{c}_t}\right) P_{s,t} F(\varphi)$$

$$+ \frac{1}{2}\Delta_{c_t} P_{s,t} F(\varphi)$$

$$- \left(\nabla V_t, \nabla P_{s,t} F(\varphi)\right)_{\dot{c}_t}$$

$$- \frac{1}{2}\left(\Delta_{\dot{c}_t} V_t\right) \cancel{P_{s,t} F(\varphi)} + \frac{1}{2}\left(\nabla V_t, \cancel{\nabla V_t}\right)_{\dot{c}_t} P_{s,t}F(\varphi)$$

$$= \underbrace{\left(\frac{1}{2}\Delta_{c_t} - (\nabla V_t, \nabla)_{\dot{c}_t}\right)}_{L_t} P_{s,t} F(\varphi)$$

$\frac{\partial}{\partial s}$-derivative - similar.

(3.27) is a gen of Prop 3.3:

$$\mathbb{E}_{\nu_0}[F] = \mathbb{E}_{\nu_t}[P_{0,t}F] \xrightarrow{\text{Gen}} \mathbb{E}_{\nu_t}[P_{s,t}F]$$

$$= \mathbb{E}_{\nu_s}[F]$$

The last statement follows by diff:

$$\frac{\partial}{\partial s}\mathbb{E}_{\nu_s}[F] = \frac{\partial}{\partial s}\mathbb{E}_{\nu_t}[P_{s,t}F]$$

$$= \mathbb{E}_{\nu_t}\left[\frac{\partial}{\partial s}P_{s,t}F\right]$$

$$= -\mathbb{E}_{\nu_t}\left[P_{s,t}L_s F\right]$$

Take $s=t$: $\frac{\partial}{\partial t}\mathbb{E}_{\nu_t}[F] = -\mathbb{E}_{\nu_t}[L_t F]$

Done.