Step initial condition:

$$\eta_x = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Then $$Q_0(x) = \begin{cases} 1 & x \leq 0 \\ \tau^x & x > 0 \end{cases}$$

$$\tilde{Q}_0(x) = \eta_x \, Q(x-1) = \begin{cases} 0 & x \leq 0 \\ \tau^{x-1} & x > 0 \end{cases}$$

$$= \frac{1-\tau}{1+\tau} \mathbb{E}\Big(Q(x_2) Q(x_1-1) \big(\tau \eta_{x_1} (1-\eta_{x_1+1}) + $$
$$+ \eta_{x_1+1} (1-\eta_{x_1}) \big) (\eta_{x_1} - \eta_{x_1+1}) \Big)$$
$$= \frac{1-\tau}{1+\tau} \mathbb{E}\Big(Q(x_2) Q(x_1-1) \cdot \big(\tau \eta_{x_1} (1-\eta_{x_1+1}) - \eta_{x_1+1} (1-\eta_{x_1}) \big) \Big)$$
$$= \frac{1-\tau}{1+\tau} \mathbb{E}\Big(Q(x_2) Q(x_1-1) \cdot \big(\tau \eta_{x_1} - \tau \eta_{x_1} \eta_{x_1+1}$$
$$- \eta_{x_1+1} + \eta_{x_1+1} \eta_{x_1} \big) =$$
$$= \frac{1-\tau}{1+\tau} \mathbb{E}\Big(Q(x_2) Q(x_1-1) \cdot \Big(\tau \frac{1-\tau^{\eta_{x_1}}}{\cancel{1-\tau}} - \frac{1-\tau^{\eta_{x_1+1}}}{\cancel{1-\tau}}$$
$$+ \cancel{(1-\tau)} \frac{(1-\tau^{\eta_{x_1+1}})(1-\tau^{\eta_{x_1}})}{\cancel{(1-\tau)^2}} \Big)$$
$$= \frac{1}{1+\tau} \mathbb{E}\Big(Q(x_2) Q(x_1-1) \cdot \big(\tau - \tau \cancel{\tau^{\eta_{x_1}}} - \cancel{1} + \cancel{\tau^{\eta_{x_1+1}}}$$
$$+ \cancel{1} - \cancel{\tau^{\eta_{x_1+1}}} - \cancel{\tau^{\eta_{x_1}}} + \tau^{\eta_{x_1+1} + \eta_{x_1}} \big)$$
$$= \frac{1}{1+\tau} \mathbb{E}\Big(Q(x_2) \cdot Q(x_1-1) \cdot \big(\tau - (1+\tau) \tau^{\eta_{x_1}} + \tau^{\eta_{x_1} + \eta_{x_1+1}} \big) \Big)$$
$$= \mathbb{E}\Big(Q(x_2) \cdot \big(p Q(x_1-1) + q Q(x_1+1) - Q(x_1) \big) \Big)$$
$$= \Delta_{x_1} \mathbb{E}\big(Q(x_2) Q(x_1) \big)$$

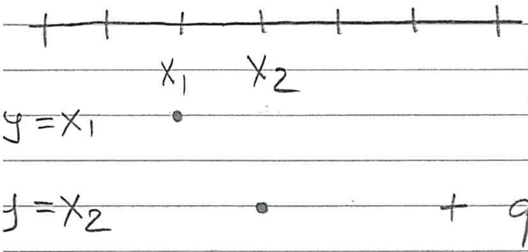
$$2 = \mathbb{E}\Big[Q(x_1) \cdot Q(x_2-1) \cdot \big(\tau^{\eta_{x_2+1}} - \tau^{\eta_{x_2}} \big) \cdot \big(p \eta_{x_2} (1-\eta_{x_2+1})$$
$$+ q (1-\eta_{x_2}) \eta_{x_2+1} \big) \Big] \underset{\text{in } 1}{\overset{\text{as}}{=}} \Delta_{x_2} \mathbb{E}\big[Q(x_2) Q(x_1) \big]$$

#

Back to Q: let us check that the ODE's we derived are satisfied on the whole of $\tilde{W} = \{x_1 < x_2 < x_3 < \dots\}$

The main interesting case is:

$$f(\eta) = Q(x_1)\,Q(x_2)$$



$$\frac{d}{dt}\mathbb{E}[f(\eta)] = \sum_{y\in\mathbb{Z}} \mathbb{E}\Big[\big(p\,\eta_y(1-\eta_{y+1}) + q(1-\eta_y)\eta_{y+1}\big)\big(Q(x_1)Q(x_2)\big|_{\eta^{y,y+1}} - Q(x_1)Q(x_2)\big|_{\eta}\big)\Big]$$

$$= \mathbb{E}\Big[\overset{①}{\big(p\,\eta_{x_1}(1-\eta_{x_1+1}) + q(1-\eta_{x_1})\eta_{x_1+1}\big)} Q(x_2)\big|_\eta \times \big(Q(x_1)\big|_{\eta^{y,y+1}} - Q(x_1)\big|_\eta\big)\Big]\Big|_{y=x_1}$$

$$+ \mathbb{E}\Big[\overset{②}{\big(p\,\eta_{x_2}(1-\eta_{x_2+1}) + q(1-\eta_{x_2})\eta_{x_2+1}\big)} Q(x_1)\big|_\eta \times \big(Q(x_2)\big|_{\eta^{y,y+1}} - Q(x_2)\big|_\eta\big)\Big]\Big|_{y=x_2}$$

$$\overset{pp3-4}{=} \Big[\big(pD_1^{-1} + qD_1 - 1\big) + \big(pD_2^{-1} + qD_2 - 1\big)\Big]\mathbb{E}[f(\eta)] \qquad (*)$$

$\#$ The last equality in more detail:

$$① = \frac{1}{(1+\tau)}\mathbb{E}\Big(\big[\tau\,\eta_{x_1}(1-\eta_{x_1+1}) + \eta_{x_1+1}(1-\eta_{x_1})\big] Q(x_2)\big|_\eta\, Q(x_1-1)\big|_\eta \big(\tau^{\eta_{x_1+1}} - \tau^{\eta_{x_1}}\big)\Big) \equiv$$

$$1 - \tau^{\eta} = \eta(1-\tau)$$

$$\tau^{\eta} = 1 - \eta(1-\tau)$$

$$\tau^{\eta_{x_1+1}} - \tau^{\eta_{x_1}} = (1-\tau)(\eta_{x_1} - \eta_{x_1+1})$$

$$\equiv p\,\eta(x)\,\eta(x)\,Q(x-1)\,Q(x-1) + q\,\eta(x+1)\,\eta(x+1)\,Q(x)\,Q(x)$$
$$-\ \eta(x)\,\eta(x+1)\,Q(x-1)\,Q(x)$$
$$= Q^2(x-1)\left(\left(p\,\eta(x) + q\,\eta(x+1)\right)\tau^{2\eta x} - \eta(x)\,\eta(x+1)\,\tau^{\eta x}\right)$$

OK, there is absolutely now reason why this should be zero either a.s. or under the sign of expectation value. (compare this with calculation on p.5) Any ideas?

For the moments of Q, the boundary condition are:

$$\left(p\left(D_{i+1}^{-1}-1\right) + q\left(D_i - 1\right)\right) u(t,\underline{x})\Big|_{x_{i+1}=x_i+1} = 0$$

$$p\,\partial_{i+1}^{(-)} u = q\,\partial_i^{(+)} u\Big|_{x_{i+1}=x_i+1}$$ "Zero flux condition"

- a Dirichlet-type B.C. There is no reason why the product moment of derivatives of $Q(=\tilde{Q})$ should obey the same B.C.'s.

Moment equations - II.

Let $Q(x) = \tau^{-N(x)}$

Let $\tilde{Q}(x) = \frac{1-D^{-1}}{\tau-1} Q(x)$

Explicitly, $\tilde{Q}(x) = \frac{Q(x)-Q(x-1)}{\tau-1}$

$$= \frac{\tau^{\eta_x}-1}{\tau-1} Q(x-1) = \eta(x) Q(x-1)$$

Equations for product moments of $\tilde{Q}$:

let $x_1 < x_2 < \ldots < x_K$

$$\tilde{u}(t, x_1 x_2 \ldots x_K) = \mathbb{E}\left[\prod_{m=1}^{K} \tilde{Q}_t(x_m)\right]$$

$$= \prod_{n=1}^{K} \frac{1-D_n^{-1}}{\tau-1} u(t, x_1 \ldots x_K)$$

as $\left[1-D_n^{-1}, \partial_t - \Delta_m\right] = 0 \ \forall n, m$,

we conclude that

$$\left(\partial_t - \sum_{m=1}^{K} \Delta_m\right) \tilde{u}(t, x_1 \ldots x_K) = 0,$$

i.e. that $\tilde{u}$'s satisfy the same equation a u's.

Boundary conditions: consider

$$\tilde{Q}(x_i)\tilde{Q}(x_{i+1}) = \eta_{x_i}\eta_{x_{i+1}} Q(x_i-1)\cdot Q(x_{i+1}-1)$$

Then

$$\left(p D_{i+1}^{-1} + q D_i - 1\right)\tilde{Q}(x_i)\,\tilde{Q}(x_{i+1})\Big|_B$$

$$= p\,\tilde{Q}(x_i)\tilde{Q}(x_{i+1}-1) + q\,\tilde{Q}(x_i+1)\tilde{Q}(x_{i+1})$$

$$- \tilde{Q}(x_i)\,\tilde{Q}(x_{i+1})\Big|_{x_{i+1}=x_i+1=x+1}$$

$$= p\,\tilde{Q}(x)\,\tilde{Q}(x) + q\,\tilde{Q}(x+1)\,\tilde{Q}(x+1)$$

$$- \tilde{Q}(x)\,\tilde{Q}(x+1) =$$

Let $\Delta_i = (pD_i^{-1} + qD_i - 1)$

Generalizing the above (and assuming that points are well separated),

$$\frac{d}{dt}u(t; x_1, x_2, \dots x_K) = \sum_{m=1}^{K} \Delta_m u(t; x_1, x_2, \dots, x_K),$$

where

Correct domain

$$\begin{cases} x_1 < x_2 \\ x_2 < x_3 \\ x_{K-1} < x_K \end{cases}$$

Origin of duality: $U_q(sl_2)$-symmetry of ASEP (Schütz, 1997)

Boundary conditions: suppose $x_{i+1} = x_i + 1$. Then

$$\tau^{N(x_i)}\tau^{N(x_{i+1})} = \tau^{N(x_i)}\tau^{N(x_i+1)} \Rightarrow$$

$$(pD_{i+1}^{-1} + qD_i - 1)\tau^{N(x_i)}\tau^{N(x_{i+1})}$$

$$= p\tau^{2N(x_i)} + q\tau^{2N(x_{i+1})} - \tau^{N(x_i)+N(x_{i+1})}$$

$$= e^{2N(x_i)}\left(p + q\tau^{2\eta_{x+1}} - \tau^{\eta_{x+1}}\right) = 0 \Rightarrow$$

$$(pD_{i+1}^{-1} + qD_i - 1)u(t,\underline{x})\Big|_{x_{i+1}=x_i+1} = 0$$

$$i = 1, 2, \dots, K-1.$$

In addition, $\lim_{x_1 \to -\infty} u(t,\underline{x}) = 0$

Is there a physical interpretation of this boundary condition?

$$\eta = \frac{1-\tau^{\eta}}{1-\tau}$$

$$= \left(p\left(1-\tau^{\eta_x}\right)\left(1-\eta_{x+1}\right) - q\left(1-\tau^{\eta_{x+1}}\right)\left(1-\eta_x\right)\right)$$

$$= \frac{1}{(1-\tau)}\left(p\left(1-\tau^{\eta_x}\right)\left(\tau^{\eta_{x+1}}-\tau\right) - q\left(1-\tau^{\eta_{x+1}}\right)\left(\tau^{\eta_x}-\tau\right)\right)$$

$$= \frac{1}{(1-\tau)}\frac{1}{1+\tau}\left(\tau\left(1-\tau^{\eta_x}\right)\left(\tau^{\eta_{x+1}}-\tau\right) - \left(1-\tau^{\eta_{x+1}}\right)\left(\tau^{\eta_x}-\tau\right)\right)$$

$$= \frac{1}{(1-\tau^2)}\Big(\tau\left(\cancel{\tau^{\eta_{x+1}}} - \tau^{\eta_x+\eta_{x+1}} - \cancel{\tau} + \cancel{\tau^{\eta_x+1}}\right)$$
$$- \left(\cancel{\tau^{\eta_x}} - \tau^{\eta_{x+1}+\eta_x} - \cancel{\tau} + \cancel{\tau^{\eta_{x+1}+1}}\right)\Big)$$

$$p = \frac{\tau}{1+\tau}$$
$$q = \frac{1}{1+\tau}$$

$$= \frac{1}{(1-\tau^2)}\left(\tau(1-\tau) + \tau^{\eta_x}\left(\tau^2-1\right) + \tau^{\eta_{x+1}+\eta_x}(1-\tau)\right)$$

$$= \frac{\tau}{1+\tau} - \tau^{\eta_x} + \frac{1}{1+\tau}\tau^{\eta_{x+1}+\eta_x} =$$

$$= p - \tau^{\eta_x} + q\,\tau^{\eta_x+\eta_{x+1}}$$ #

$$= \mathbb{E}\left[\tau^{N_t(x-1)}\left(p - \tau^{\eta_x} + q\,\tau^{\eta_x+\eta_{x+1}}\right)\right]$$

$$= \mathbb{E}\left[p\,\tau^{N_t(x-1)} - \tau^{N_t(x)} + q\,\tau^{N_t(x+1)}\right]$$

$$= \left(\underbrace{p D^{-1} + q D - 1}_{\Delta}\right) u(t,x)$$

$f: \mathbb{Z} \to \mathbb{R}$, $Df(z) = f(z+1)$, $D^{-1}f(z) = f(z-1)$

#

Moment equations - I

Let $N_t(x) = \sum_{y=-\infty}^{x} \eta_t(y)$ (a.s. finite)

Let

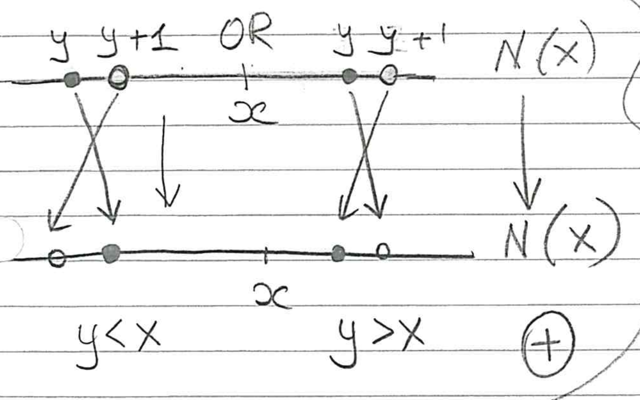
$$u(t, x_1 x_2 \dots x_K) = \mathbb{E}\left[\prod_{m=1}^{K} \tau^{N_t(x_m)}\right],$$

where $\tau = \frac{p}{q} < 1$

K=1

$$\frac{du}{dt}(t,x) = \mathbb{E}\left[L\left(\tau^{N_t(x)}\big|_\eta\right)\right]$$

$$= \sum_{y\in\mathbb{Z}} \mathbb{E}\left[\left(p\eta_y(1-\eta_{y+1}) + q(1-\eta_y)\eta_{y+1}\right)\left(\tau^{N_t(x)}\big|_{\eta^{y,y+1}} - \tau^{N_t(x)}\big|_\eta\right)\right] = \mathbb{E}\left[\left(p\eta_x(1-\eta_{x+1}) + q(1-\eta_x)\eta_{x+1}\right)\left(\tau^{N_t(x)}\big|_{\eta^{x,x+1}} - \tau^{N_t(x)}\big|_\eta\right)\right] =$$



$$= \mathbb{E}\left[\tau^{N_t(x-1)} Z\right] =$$

$$\# \quad Z = \left(\tau^{\eta_x}\big|_{\eta^{x,x+1}} - \tau^{\eta_x}\big|_\eta\right)\left(p\,\underbrace{\eta_x(1-\eta_{x+1})}_{\eta_x=1\ \eta_{x+1}=0} + q\,\underbrace{\eta_{x+1}(1-\eta_x)}_{\eta_{x+1}=1\ \eta_x=0}\right)$$

$$= (1-\tau)\,p\,\eta_x(1-\eta_{x+1}) + (\tau-1)\,q\,\eta_{x+1}(1-\eta_x)$$

$$= (1-\tau)\left(p\,\eta_x(1-\eta_{x+1}) - q\,\eta_{x+1}(1-\eta_x)\right) =$$

$$= \sum_{y} \left[p\eta'_y (1-\eta'_{y+1}) + q\eta'_{y+1} (1-\eta'_y) \right] f(\eta'^{y,y+1})$$
$$- \sum_{y} \left[p\eta'_y (1-\eta'_{y+1}) + q\eta'_{y+1} (1-\eta'_y) \right] f(\eta')$$
$$= \sum_{y} \left[p\eta'_y (1-\eta'_{y+1}) + q\eta'_{y+1} (1-\eta'_y) \right] \left[f(\eta'^{y,y+1}) - f(\eta') \right]$$

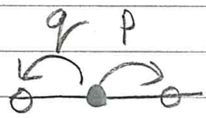
#

Initial conditions used:

at most finite number of particles to the left of the origin. (I.e. there is a left-most particle)
Then $\forall t>0$ the left most particle exists a.s.

KPZ reading group — Jan-Feb 2014

ASEP.



1 The model

$$\eta(t) = \{\eta_x(t)\}_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} \text{ - configuration space}$$

$$\eta_x(t) = \begin{cases} 1 & \text{particle at site } x \text{ at time } t \\ 0 & \text{hole at site } x \text{ at time } t \end{cases}$$

Dynamics: $\eta \to \eta^{y,y+1}$ at rate $p\,\eta_y(1-\eta_{y+1})$

$\eta \to \eta^{y,y+1}$ at rate $q(1-\eta_y)\eta_{y+1}$

$p+q=1$

If $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ is a function on config space,

ASEP generator is

(1) $$Lf(\eta) = \sum_{y\in\mathbb{Z}} \left[p\,\eta_y(1-\eta_{y+1}) + q(1-\eta_y)\eta_{y+1}\right]\left(f(\eta^{y,y+1}) - f(\eta)\right)$$

\# $$\Pr(\eta_{t+\varepsilon} = \eta \mid \eta_t = \eta')$$

$$= +\sum_y \chi(\eta' = \eta^{y,y+1})\left[p\,\eta'_y(1-\eta'_{y+1}) + q\,\eta'_{y+1}(1-\eta'_y)\right]\varepsilon$$

$$+ \delta_{\eta',\eta}\left(1 - \varepsilon\sum_y\left[p\,\eta'_y(1-\eta'_{y+1}) + q\,\eta'_{y+1}(1-\eta'_y)\right]\right)$$

$\Rightarrow$ $Lf(\eta') =$

$$= \sum_\eta\sum_y \chi(\eta' = \eta^{y,y+1})\left[p\,\eta'_y(1-\eta'_{y+1}) + q\,\eta'_{y+1}(1-\eta'_y)\right]f(\eta)$$

$$-\varepsilon\sum_\eta \delta(\eta,\eta')\sum_y\left(p\,\eta'_y(1-\eta'_{y+1}) + q\,\eta'_{y+1}(1-\eta'_y)\right)f(\eta)$$