COMPACTNESS, LARGE DEVIATIONS AND A RIGOROUS THEORY OF POLARON

In a reasonable topological space, large deviation estimates essentially deal with probabilities of events that are asymptotically (exponentially) small, and in a certain sense, quantify the rate of these decaying probabilities. In such estimates, upper bounds for such small probabilities often require compactness of the ambient space, which is often absent in problems arising in statistical mechanics (for example, distributions of empirical measures of simple random walks or that of Brownian motion in the full space $\mathbb{R}^d$). Motivated by such a problem, we present a robust theory of "translation-invariant compactification" of probability measures in $\mathbb{R}^d$. Thanks to an inherent shift-invariance of the underlying problem, we are able to apply this abstract theory painlessly and solve some long standing open problems in quantum mechanics. In particular, we will give applications to the following questions:

• We will treat the case of the Polaron measure, written as

$$\mathrm{d}\widehat{\mathbb{P}}_{\alpha,T} = \frac{1}{Z_{\alpha,T}} \exp\left\{\alpha \int_0^T \int_0^T \frac{\mathrm{e}^{-|t-s|}}{|\omega(t)-\omega(S)|} \mathrm{d}s\,\mathrm{d}t\right\} \mathrm{d}\mathbb{P} \tag{0.1}$$

and show that, as $T \to \infty$ and fixed α, $\mathrm{d}\widehat{\mathbb{P}}_{\alpha,T}$ converges an an *explicit mixture of Gaussian measures*. As a corollary, we will deduce a central limit theorem for the distribution of

$$\mathbb{P}_{\alpha,T}\left(\frac{\omega(T)-\omega(-T)}{\sqrt{2}T} \in \cdot\right) \to N(0, \sigma^2(\alpha)),$$

where the variance $\sigma^2(\alpha) = m_{\mathrm{eff}}^{-1}(\alpha)$ is explicit and is the inverse of the *effective mass* of the Polaron.

• We will rigorously analyze the mean-field Polaron measure

$$\mathrm{d}\widehat{\mathbb{P}}_T^{\mathrm{mf}} = \frac{1}{Z_T} \exp\left\{\frac{1}{T} \int_0^T \int_0^T \frac{1}{|\omega(t)-\omega(S)|} \mathrm{d}s\,\mathrm{d}t\right\} \mathrm{d}\mathbb{P} \tag{0.2}$$

and show that as $T \to \infty$ it converges to a mixture of diffusion processes, called the *Pekar process*, with generator $\frac{1}{2}\Delta + \left(\frac{\nabla\phi}{\phi}\right) \cdot \nabla$, where ϕ is a rotationally symmetric solution of the Pekar variational formula $\sup_\phi\{\int\int \frac{\phi^2(x)\phi^2(y)\mathrm{d}x\mathrm{d}y}{|x-y|} - \frac{1}{2}\|\nabla\phi\|_2^2\}$.

• Finally, we will show that in the *strong coupling limit* $\alpha \to \infty$, the distributions of the rescaled Polaron limit $\lim_{T\to\infty} \widehat{\mathbb{P}}_{\alpha,T}$ and that of the mean-field Polaron measures $\lim_{T\to\infty} \widehat{\mathbb{P}}_T^{\mathrm{mf}}$ in fact coincide, the limit being identified as the increments of the Pekar process.

These results rigorously prove conjectures made by Spohn in 1980s and are based on a series of results obtained in collaboration with S. R. S. Varadhan (New York), as well as with Erwin Bolthausen (Zurich) and Wolfgang Koenig (Berlin).