

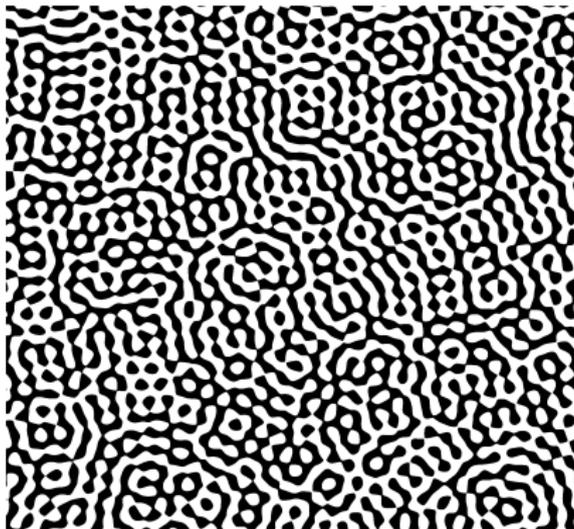
The phase transition for planar Gaussian percolation without FKG

Stephen Muirhead (QMUL)

Alejandro Rivera (EPFL)

Hugo Vanneuville (ETH Zurich)

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Level set percolation of Gaussian fields

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth planar Gaussian field.

Suppose f is centred and stationary, so that it is uniquely defined by its covariance kernel

$$\kappa(x) := \mathbb{E}[f(0)f(x)].$$

We study the ‘level set percolation’ associated to f , i.e. we consider whether the excursion and level sets

$$\{f \leq \ell\} := \{x \in \mathbb{R}^2 : f(x) \leq \ell\}$$

$$\{f = \ell\} := \{x \in \mathbb{R}^2 : f(x) = \ell\}$$

have unbounded connected components.

By monotonicity, one can define a *critical level* via

$$\begin{aligned}\theta(\ell) &:= \mathbb{P}(\{f \leq \ell\} \text{ has an unbounded connected component}), \\ \ell_{\text{crit}} &:= \inf\{\ell : \theta(\ell) > 0\} \in [-\infty, \infty].\end{aligned}$$

The excursion sets at $\ell = 0$ are 'self-dual' ($\{f \leq 0\} \stackrel{d}{=} \{f \geq 0\}$), so it is natural to predict that $\ell_{\text{crit}} = 0$:

Conjecture (Dykhne '70, Zallen & Scher '71)

If f is ergodic (and perhaps some other mild conditions) then

$$\ell_{\text{crit}} = 0.$$

More precisely

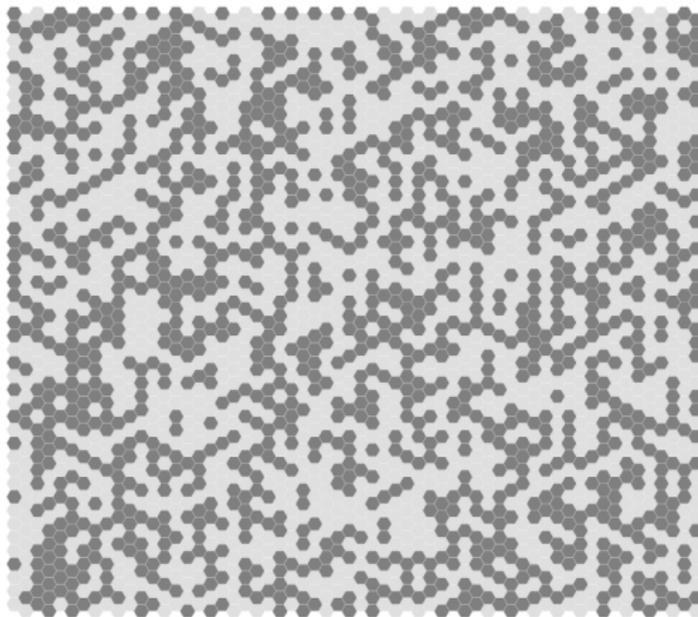
- (1) *If $\ell \leq 0$, $\{f \leq \ell\}$ has bounded connected components a.s. (equiv. $\{f = \ell\}$ has bounded components a.s. for all ℓ)*
- (2) *If $\ell > 0$, $\{f \leq \ell\}$ has a unique unbounded conn. comp. a.s.*

We can compare to other 'self-dual' planar models:

1) Boolean percolation on the hexagonal lattice

Critical point = Self-dual point ($p_{\text{crit}} = 1/2$)

[Harris '61, Kesten '82]

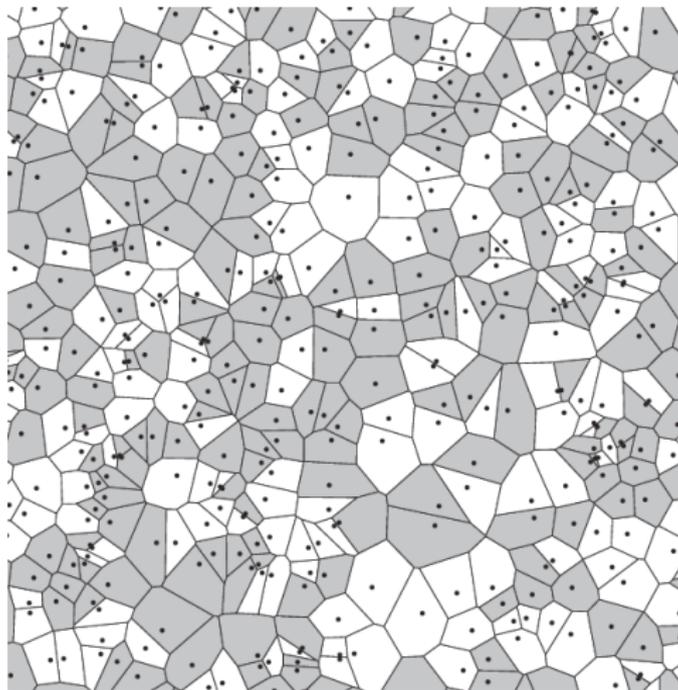


Credit: Wikipedia

2) Poisson Voronoi percolation

Critical point = Self-dual point ($p_{\text{crit}} = 1/2$)

[Bollobás & Riordan '06]

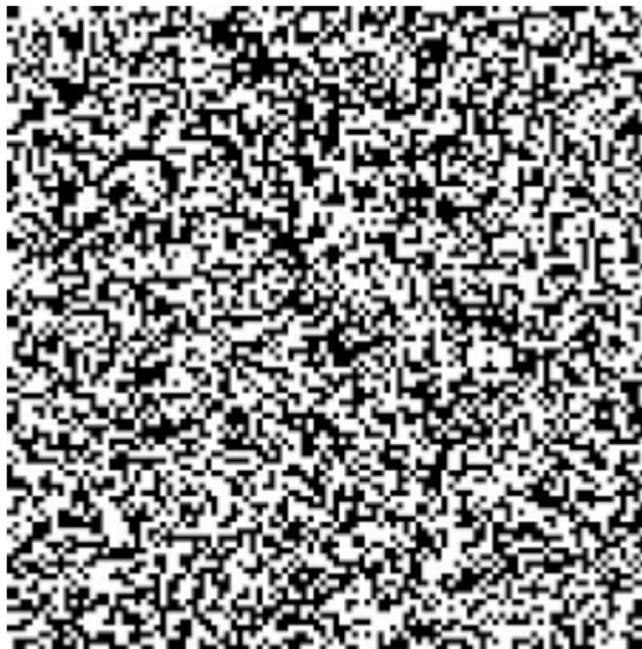


Credit: Bollobás & Riordan

3) Ising model on the square lattice

Critical point = Self-dual point ($\beta_{\text{crit}} = \log(1 + \sqrt{2})$)

[Kramers & Wannier '41, Onsager '44]



Credit: Wikipedia

Example: The random plane wave

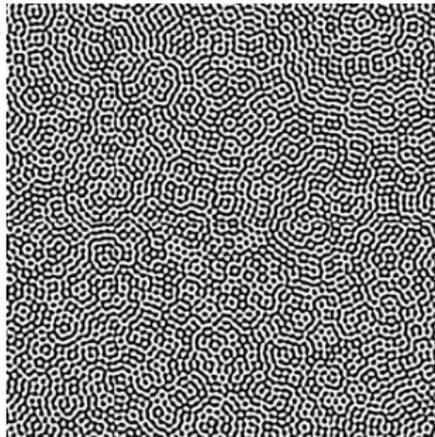
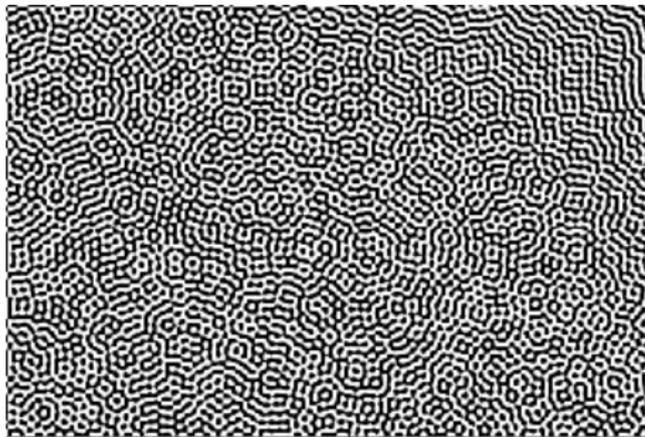
The **random plane wave** (RPW) is the stationary planar Gaussian field with covariance $\kappa(x) = J_0(|x|)$, where J_0 is the zeroth Bessel function

$$J_0(r) = \sqrt{\frac{2}{\pi r}} \cos(r - \pi/4) + O(1/r).$$

The RPW has spectral measure that is uniform on the unit circle, and so can be considered the isotropic 'Gaussian object' in the space of functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ that satisfy the planar **Helmholtz equation**

$$\Delta f = -f.$$

Berry's conjecture [Berry '77] states that the RPW 'models' generic eigenfunctions of the Laplacian on chaotic domains/manifolds.

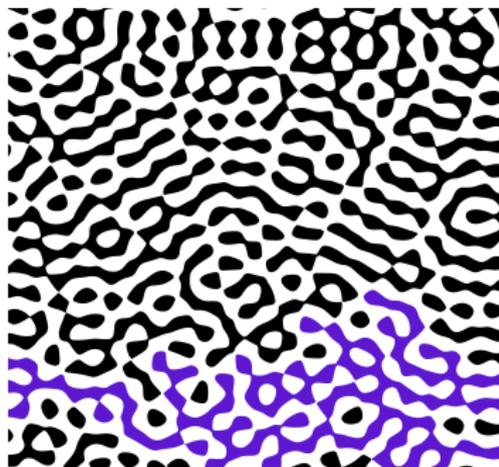


A Dirichlet eigenfunction of a stadium (left) and a realisation of the RPW with the same energy (right). Credit: Bogomolny & Schmit.

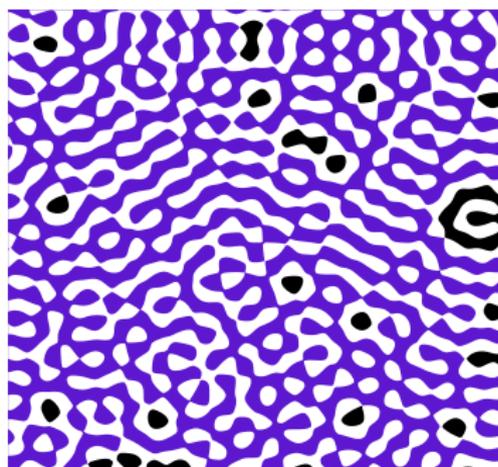


Conjecture (Dykhne '70, Zallen & Scher '71, Sodin '12)

For the RPW, $\ell_{crit} = 0$.



$\ell = -0.1$



$\ell = 0.1$

Previous results

Recently $\ell_{\text{crit}} = 0$ has been proven for a class of Gaussian fields that satisfy two important assumptions:

1) Positive associations (FKG)

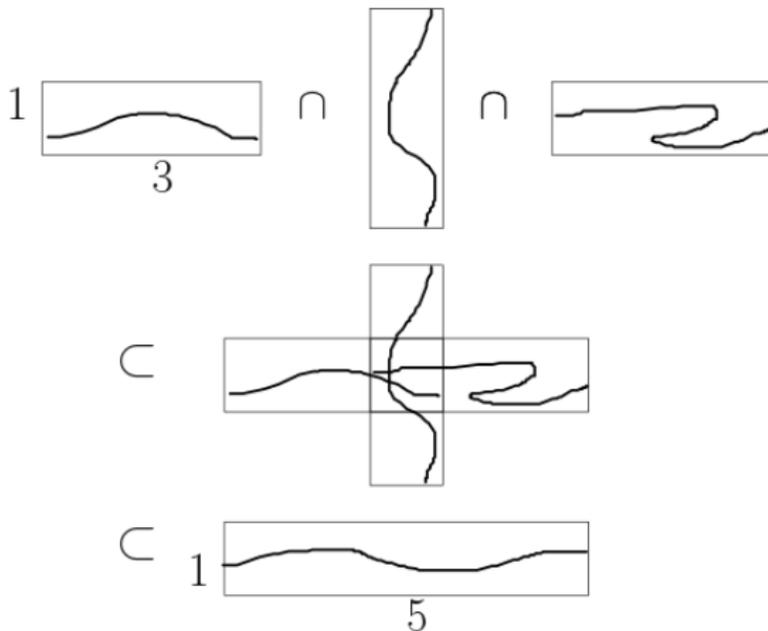
If A and B are increasing (i.e. $f \in A, g \geq 0 \implies f + g \in A$), then

$$\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B].$$

For Gaussian fields, (FKG) is equivalent to pointwise positive correlations ($\kappa(x) \geq 0$ for all $x \in \mathbb{R}^2$) [Pitt '82].

(FKG) is an essential tool in classical percolation theory.

For instance, it allows for 'gluing constructions' such as:



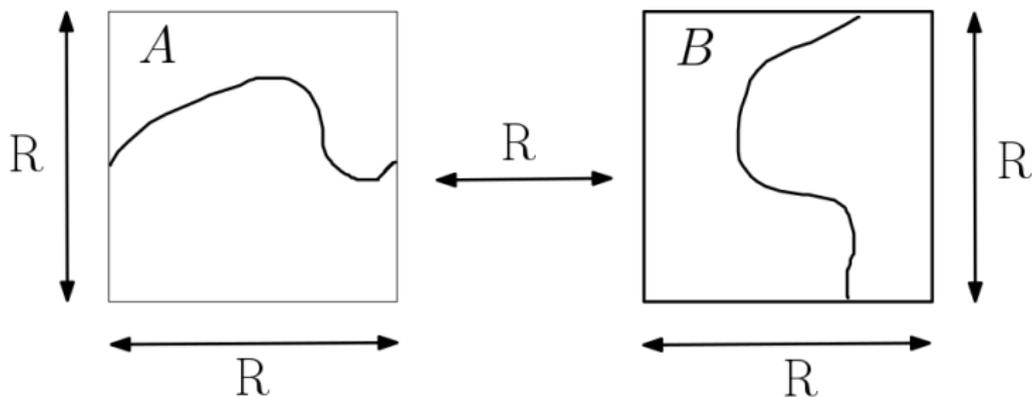
So

$$\mathbb{P}[\text{Cross a } 5 \times 1 \text{ rectangle}] \geq \mathbb{P}[\text{Cross a } 3 \times 1 \text{ rectangle}]^3.$$

2) Short range correlations (SRC)

The covariance kernel κ is absolutely integrable at infinity ($\kappa \in L^1$).

This allows the 'decoupling' of crossing events on far away domains (known as *quasi-independence*):



$$|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]| \xrightarrow{R \rightarrow \infty} 0.$$

Previous results

The first field for which $l_{\text{crit}} = 0$ was confirmed was the *massive GFF* on \mathbb{Z}^2 [Rodriguez '16] (which as well as (FKG) and (SRC) has a lot of extra structure).

More recently, $l_{\text{crit}} = 0$ has been proven completely under (FKG)–(SRC) and mild extra conditions [Rivera & Vanneuille '19, M. & Vanneuille '20, Garban & Vanneuille '19, Rivera '19].

If only one of (FKG) or (SRC) are satisfied there are partial results:

(FKG) $\implies l_{\text{crit}} \geq 0$ [Alexander '96, Gandolfi, Keane & Russo '88]

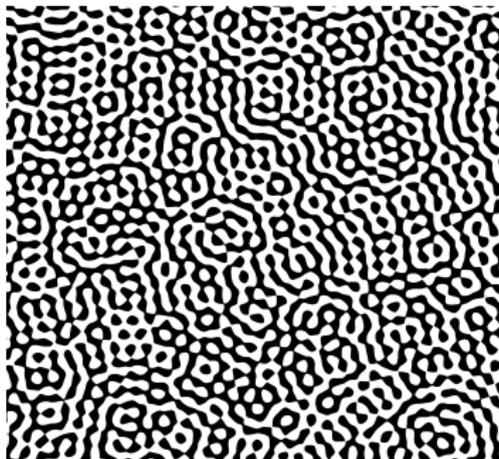
(SRC) $\implies l_{\text{crit}} < \infty$ [Molchanov & Stepanov '83]

and $l_{\text{crit}} \geq 0$ in a perturbative non-FKG regime [Beffara & Gayet '18]

Recall that the covariance of the RPW is the radial Bessel function

$$J_0(r) = \sqrt{\frac{2}{\pi r}} \cos(r - \pi/4) + O(1/r).$$

Hence the RPW has neither (FKG) nor (SRC), so nothing was known about ℓ_{crit} (not even that it was finite).



Our result

We establish $\ell_{\text{crit}} = 0$ for a very wide class of Gaussian fields without assuming (FKG)–(SRC), including the RPW.

Let us introduce some mild assumptions:

Assumptions

1. (*Smoothness*) f is a.s. C^3 -smooth.
2. (*Non-degeneracy*) The Gaussian random vector $(f(x), \nabla f(x), f(y), \nabla f(y))$ is non-degenerate for $x \neq y$.
3. (*Covariance decay*) $\kappa(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

We also recall two common notions of symmetry:

1. f is **D_4 -symmetric** if it is symmetric wrt reflections in the horizontal/vertical axes and rotations by $\pi/2$.
2. f is **isotropic** if it is symmetric wrt all rotations.

Our main result is the following:

Theorem (M., Rivera & Vanneuville)

Let f satisfy the above mild assumptions.

- ▶ *Suppose that f is D_4 -symmetric. Then $\ell_{\text{crit}} \geq 0$ (equiv. all non-zero level lines are bounded).*
- ▶ *Suppose in addition that f is isotropic, and there exists a $\delta > 0$ such that, as $|x| \rightarrow \infty$,*

$$|\kappa(x)|(\log \log(|x|))^{2+\delta} \rightarrow 0.$$

Then $\ell_c = 0$. In particular $\ell_c = 0$ for the RPW.

Open questions

Question 1: Do our techniques apply to other non-FKG models?

There are many important non-FKG models in percolation / statistical physics:

- ▶ Anti-ferromagnetic Ising models
- ▶ Certain regimes of the random cluster and $O(N)$ loop models
- ▶ Boolean / Voronoi models on non-Poisson point processes
- ▶ ...

These models are generally poorly understood; in fact, to our knowledge, ours is **the first rigorous computation of a critical point for a non-FKG model** that is neither integrable nor in a perturbative regime.

We hope that our techniques are more widely applicable since they rely only on:

1. **Planarity**
2. **Ergodicity / weak spatial independence**
3. **Symmetry**
4. **Level set representation / monotonic coupling**
5. **Gaussianity**

Question 2: Are the assumptions in the theorem necessary?

- ▶ Is D_4 -symmetry sufficient? Is any symmetry needed?
- ▶ Is $\kappa(x) \rightarrow 0$ sufficient? What about ergodicity?
- ▶ Can C^3 -smoothness be replaced with Hölder continuity?

It would be very nice to remove isotropy, since then the techniques apply immediately to discrete (Gaussian) models on \mathbb{Z}^2 .

We only know how to remove it if correlations decay sufficiently rapidly ((SRC) should be enough).

Question 3: What happens at criticality ($\ell = 0$)? Are zero level lines bounded?

This is the analogue of the ‘no percolation at criticality conjecture ($\theta(p_c) = 0$)’, and it seems very challenging (it is still open for Bernoulli percolation on \mathbb{Z}^3 !).

It is known that there are no unbounded zero level lines under (FKG) [Alexander ‘96, Gandolfi, Keane & Russo ‘88].

Without (FKG), it has been proven in a perturbative regime (which does not cover the RPW) [Beffara & Gayet ‘18].

Question 4: Is there exponential decay of connectivity in the subcritical phase, i.e. if $\ell < 0$ then is

$$\mathbb{P}[0 \text{ is connected to } \partial B(R) \text{ in } \{f \leq \ell\}] \leq e^{-cR}?$$

If κ decays sufficiently rapidly (again (SRC) should be enough) then we can deduce this.

In general our techniques only give that crossing probabilities at scale R decay as

$$e^{-c\sqrt{\min\{\log R, 1/\bar{\kappa}(\sqrt{R})\}}}$$

where $\bar{\kappa}(R) = \sup_{|x| \geq R} |\kappa(x)|$.

In particular, for the RPW the bound is

$$e^{-c\sqrt{\log R}}.$$

Question 5: Do the zero level lines have a conformal invariant scaling limit?

The RPW has been conjectured to lie in the 'Bernoulli percolation universality class' [Bogomolny & Schmit '07, Weinrib '84], and so in particular the zero levels lines should converge to $CLE(6)$.

This is supported by numerical evidence [Bogomolny & Schmit '07, Beliaev & Kerata '13].

According to 'Harris criterion', this should be true whenever

$$R^{-5/2} \int_{B(R) \times B(R)} \kappa(x-y) dx dy \rightarrow 0.$$

In general this requires $\kappa(R) \ll R^{-3/2}$ but is also satisfied for the RPW due to cancellations when integrating over $J_0(x)$.

Outline of the proof

First let us recall the classical approach [Kesten '82, Bollobás & Riordan '06, Beffara & Duminil-Copin '12] to proving that the self-dual point of a planar model is critical:

Step 1: Use self-duality to establish properties of the self-dual point ('box-crossing estimates' = 'RSW theory').

Step 2: Use these properties and a differential inequality (e.g. BKKKL, OSSS etc) to show that there is a sharp threshold at the self-dual point.

Step 3: Deduce that the self-dual point is critical.

We cannot carry out this procedure, since the RSW theory in Step 1 relies crucially on FKG.

Instead we **reverse the order of the first two steps**:

Step 1: Prove a sharp threshold result for crossing events on large scales **without identifying the threshold point**.

For this we cannot use the differential inequalities mentioned above since they (implicitly) rely on FKG.

We use a new technique based on Chatterjee's theory of superconcentration (similar to BKKKL, but with important differences).

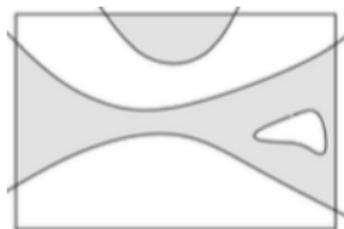
Step 2: Inject the sharp threshold result into the RSW theory to deduce crossings of rectangles with high probability at levels $\ell > 0$.

Step 3: Deduce that the self-dual point is critical.

Step 1: Sharp thresholds

The new sharp threshold result is based on properties of the **threshold** associated to a crossing event.

Consider a quad (=‘topological rectangle’) $Q \subset \mathbb{R}^2$, and the event $\text{Cross}_\ell(Q)$ that the excursion set $\{f \leq \ell\}$ crosses the quad from left to right.

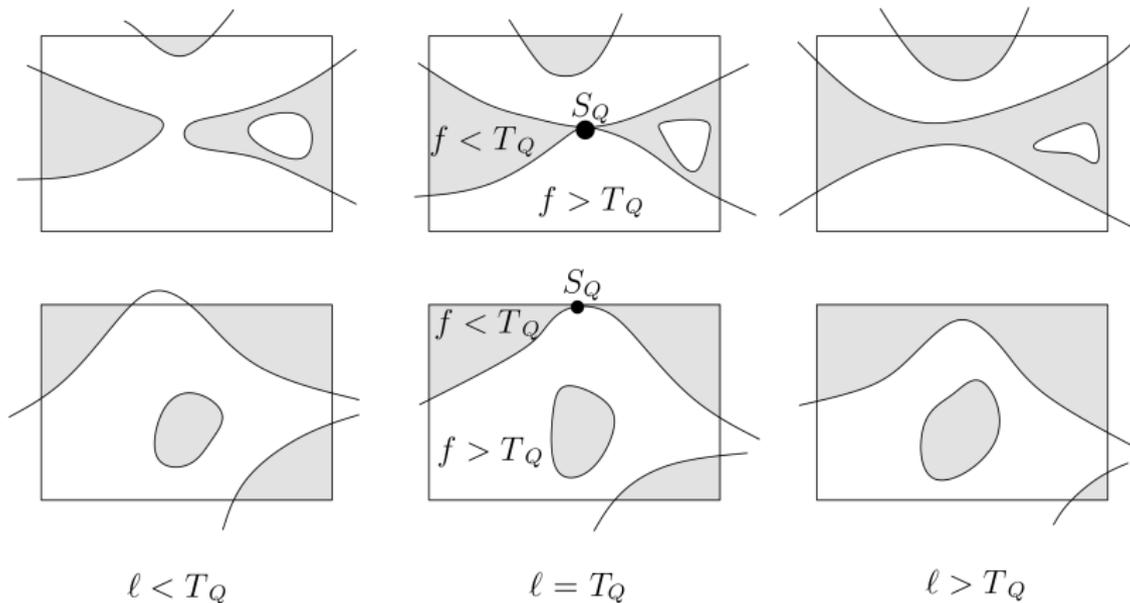


Since $\text{Cross}_\ell(Q)$ is increasing in ℓ , there exists a (random) threshold $T_Q \in [-\infty, \infty]$ such that:

1. $\text{Cross}_\ell(Q)$ does not occur for $\ell < T_Q$;
2. $\text{Cross}_\ell(Q)$ does occur for $\ell > T_Q$.

Since the realisations of f are 'nice', we can also associate to the quad Q a unique **threshold saddle point** $S_Q \in Q$ which satisfies

$$f(S_Q) = T_Q.$$



We consider the concentration properties of the random vector

$$(S_Q, T_Q) \in Q \times \mathbb{R}.$$

Our sharp threshold theorem says roughly:

T_Q is concentrated \iff the density of S_Q is uniformly small.

It is based on an argument of Chatterjee, who observed that the 'superconcentration' of the maximum of a Gaussian vector is related to the delocalisation of the argmax.

Let us state a precise version of (one direction of) this theorem:

Theorem (M., Rivera & Vanneuville)

There exists a constant $c = c(\kappa) > 0$ such that, for each quad Q ,

$$\text{Var}(T_Q) \leq c \inf_{r>0} \max \{ \bar{\kappa}(r), |\log(\sigma_Q(4r))|^{-1} \},$$

where

$$\bar{\kappa}(r) = \sup_{|x| \geq r} |\kappa(x)|$$

and

$$\sigma_Q(r) = \sup_{x \in \mathbb{R}^2} \mathbb{P}(S_Q \in B_x(r))$$

is the maximal probability over balls of radius r that the threshold saddle S_Q lies in this ball.

The threshold theorem is useful because one can prove, using only soft properties (ergodicity, D_4 -symmetry), that the threshold saddle S_Q delocalises as the scale of Q grows.

This is essentially a consequence of two facts:

1. (Delocalisation in the bulk) The field f has no saddle points whose four 'arms' connect to infinity.
2. (Delocalisation on the boundary) If $H = \{(x, y) : y \geq 0\}$ is a half-plane, then the field $f|_H$ has no unbounded level lines that intersect ∂H .

The former can be proven with a Burton-Keane type argument (which appeared earlier in [Beliaev, McAuley & M. '19]), and the latter with an argument due to Harris.

The upshot is that we prove crossing events on large scales have a sharp threshold: For each quad Q ,

$$\text{Var}(T_{RQ}) \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

However, the ‘value’ of the threshold $\mathbb{E}[T_{RQ}]$ may depend on the quad Q and scale R .

The exception is for squares, for which D_4 -symmetry implies that, if Q is a unit square, for all $R > 0$

$$\mathbb{E}[T_{RQ}] = 0.$$

This gives an immediate sharp threshold result for squares:

For $\ell < 0$, $\mathbb{P}(\text{Cross}_\ell(R \times R \text{ square})) \rightarrow 0$, as $R \rightarrow \infty$

For $\ell > 0$, $\mathbb{P}(\text{Cross}_\ell(R \times R \text{ square})) \rightarrow 1$, as $R \rightarrow \infty$

Step 2: Injecting sharp thresholds into the proof of RSW

'RSW theory' refers to arguments of the form:

Squares are crossed with non-negligible probability

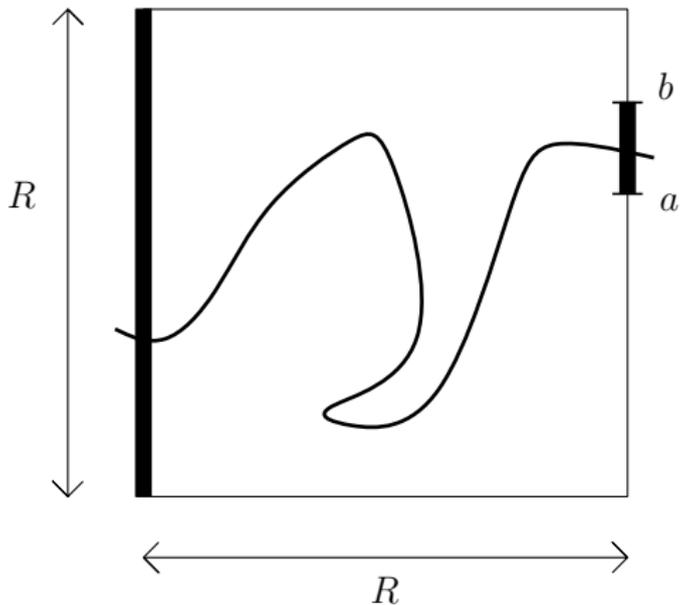
\implies Rectangles are crossed with non-negligible probability.

Although RSW theory relies crucially on FKG, by combining geometric constructions used in RSW theory with the sharp threshold theorem, we prove a 'sprinkled' version of RSW:

Squares are crossed with non-negligible probability at $\ell = 0$

\implies Rectangles are crossed with high probability at $\ell > 0$.

The 'building blocks' of the construction are quads $\mathcal{H}_R(a, b)$ of the form:



Proposition (Tassion '16)

There exists a sequence $R_n \rightarrow \infty$ and, for each $n \in \mathbb{N}$, a finite set of quads $\{Q_i^n\}_i$ that are translations, reflections in the horizontal/vertical axes, and $\pi/2$ rotations, of quads contained in

$$\{\mathcal{H}_r(a, b) : r \in \{2R_n/3, R_n\}\},$$

such that the following holds for each $n \in \mathbb{N}$:

1. For all $\ell \in \mathbb{R}$,

$$\bigcap_i \text{Cross}_\ell(Q_i^n) \subset \text{Cross}_\ell(2R_n \times R_n \text{ rectangle})$$

2. $\mathbb{P}(\text{Cross}_0(Q_i^n)) > 1/16$

Lemma

In choosing Q_i^n , we can demand that $b - a \geq \beta_{R_n} \gg 1$.

The proof of $\ell_{\text{crit}} \geq 0$ is now straightforward:

By the threshold theorem, all quads in the collection

$$\{\mathcal{H}_r(a, b) : r \in \{2R/3, R\}, b - a \geq \beta_R\}$$

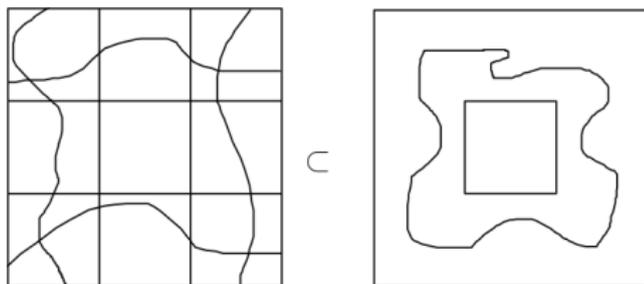
have (uniformly) sharp thresholds as $R \rightarrow \infty$.

By the geometric construction, and the union bound (which replaces FKG!), we deduce that, for $\ell > 0$,

$$\limsup_{R \rightarrow \infty} \text{Cross}_\ell(2R \times R \text{ rectangle}) = 1.$$

At this point we can conclude with classical arguments:

1. By gluing



we have that annulus circuits in $\{f \leq \ell\}$ occur with high probability at scales $R_n \rightarrow \infty$.

2. Hence

$$\bigcap_R \{B(R) \text{ is surrounded by a circuit in } \{f \leq \ell\}\}$$

occurs with high probability, and so by ergodicity it occurs a.s.

3. Hence $\{f > \ell\}$ has only bounded components, and so by symmetry the same is true for $\{f \leq \ell\}$, $\ell < 0$. Thus $\ell_{\text{crit}} \geq 0$.

The proof of $\ell_{\text{crit}} \leq 0$ is more involved because in order to argue that

$$\begin{aligned} & \limsup_{R \rightarrow \infty} \text{Cross}_\ell(2R \times R \text{ rectangle}) \rightarrow 1 \quad (1) \\ \implies & \{f \leq \ell\} \text{ has an unbounded connected component} \end{aligned}$$

we must make the convergence in (1) quantitative.

First, we need quantitative delocalisation of the threshold saddle. Under the assumption of isotropy, we can deduce this for annular circuits (by symmetry), and that is enough.

Second, we need to control the subsequence on which (1) occurs. Here we use the recent improvements to the RSW theory due to Köhler-Schindler & Tassion.

Finally, we need a 'large deviation' version of the sharp threshold theorem, borrowing an idea of Tanguy [Tanguy '15].

Thank you!