Random models for slow-fast dynamics in absence of ergodicity

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ODEs and Random Processes

- ODEs define deterministic dynamics, sometimes chaotic
  hyperbolic dynamics $\Longrightarrow$ random process (symbolic dynamics)
- slow-fast system of ODEs:

$$\dot{x} = \epsilon S(x, y) \qquad \text{slow equations}$$
$$\dot{y} = F(x, y) \qquad \text{fast equations}$$

- assuming existence of an invariant measure μ and ergodicity for the fast system
  Ergodic averaging (Anosov 1962): slow component of a solution

$$x \underset{\epsilon \to 0}{\longrightarrow} \bar{x} \qquad \text{(in probability for } |t] \leq \tfrac{T}{\epsilon}\text{)}$$

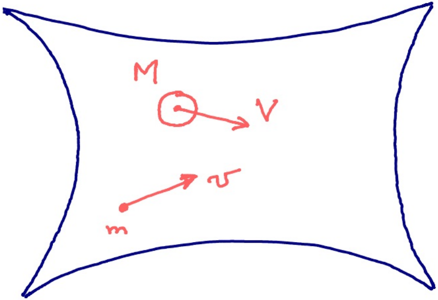
  to a solution of the averaged equation

$$\dot{\bar{x}} = \epsilon \bar{S}(\bar{x}) \qquad \bar{S}(x) := \int S(x, y)\, d\mu(y).$$

- theory of large deviations and modelling of slow dynamics on long time scales are popular topics in Dynamical System and are closely related to random processes.

Example: Brownian motion

A small heavy disk and a point particle in a 2^d container



free motion + elastic collisions

mass ratio $\epsilon^2 = \frac{m}{M} \ll 1$

$V_0 \approx \epsilon v_0$ initial velocities

Thm (Chernov, Dolgopyat 2009: Brownian Brownian motion)
On a time interval between collisions of the disk with the container walls, the speed of the disk weakly converges to a two-dimensional Brownian motion (if properly normalised).

(Disclaimer: important technical details are dropped)

Ergodicity and Energy equipartition

Hamiltonian dynamics: Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$ defines Hamiltonian equations

$$\dot{q}_k = \partial_{p_k} H(q,p), \qquad \dot{p}_k = -\partial_{q_k} H(q,p), \qquad k = 1,\ldots,n$$

Energy conservation: $M_c = \{(q,p) : H(q,p) = c\}$ is an invariant set.

Invariant (Liouville) measure μ_c on M_c (microcanonical ensemble)

Ergodicity: the time average converges to the space average (a.e.)

$$\frac{1}{T}\int_0^T f\big(q(t),p(t)\big)\,dt \underset{T\to\infty}{\longrightarrow} \langle f\rangle_c = \frac{1}{\mu_c(M_c)}\int_{M_c} f\,d\mu_c$$

Example (Interacting particles in a potential): q_k is coordinate of a particle, p_k is its momentum, m_k is its mass.

$$H = \sum_{k=1}^{n} \frac{p_k^2}{2m_k} + U(q_1,\ldots,q_n)$$

Space average is explicit: $\left\langle \frac{p_k^2}{2m_k} \right\rangle_c = \frac{V(c)}{2V'(c)}$ where $V(c) = \mathrm{vol}\{H < c\}$

Energy equipartition law, provided ergodic on M_c

Ergodicity and adiabatic laws

Slow-fast Hamiltonian dynamics: Hamiltonian function $H : \mathbb{R}^{2n+2m} \to \mathbb{R}$ defines Hamiltonian equations

$$\begin{array}{lll} \dot{q}_k = \partial_{p_k} H, & \dot{p}_k = -\partial_{q_k} H, & k = 1, \ldots, n \\ \dot{u}_k = \epsilon \partial_{v_k} H, & \dot{v}_k = -\epsilon \partial_{u_k} H, & j = 1, \ldots, m \end{array}$$

$\epsilon \ll 1$ small parameter

Conservation of the energy, invariant measure

The ergodic averaging theorem has a corollary:

Ergodicity of the fast system $\Rightarrow$ **adiabatic invariant**:

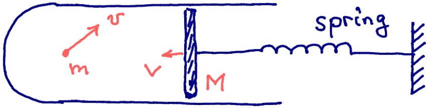
$$V(u, v, c) = \mathrm{vol}\{ (x, y) : H(q, p, u, v) < c \}$$

its time evolution converges to a constant (in probability for $|t| < \frac{T}{\epsilon}$)

Remark: If V was preserved exactly, the full system would be non-ergodic on the energy level $H = c$.

Example: springy billiard in a stadium

A point particle in a container with a piston supported by a spring



free motion + elastic collisions

mass ratio $\epsilon^2 = \dfrac{m}{M} \ll 1$, $\frac{v}{V} \ll 1$.

piston position: x $\qquad$ $A(x)$ area inside billiard

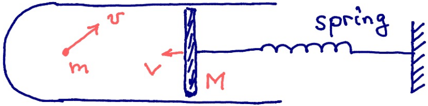
$$H = \underbrace{\frac{y^2}{2M} + \frac{kx^2}{2}}_{E_w} + \underbrace{\frac{p_1^2}{2m} + \frac{p_2^2}{2m}}_{E_p}$$

Ergodic averaging theory: $J = E_p^{d/2} A(x)$ is an adiabatic invariant
Effective Hamiltonian for the motion of the piston:

$$H_{\mathrm{eff}}(x, y) = \underbrace{\frac{y^2}{2M} + \frac{kx^2}{2}}_{E_w} + \underbrace{\left(\frac{J}{A(x)}\right)^{2/d}}_{\text{effective potential}}$$

The motion of the piston $\underset{\epsilon \to 0}{\longrightarrow}$ a periodic function (for $|t| < \frac{T}{\epsilon}$)

Springy billiard in a stadium: two sides of ergodicity



if the system is ergodic, then the equipartition law implies that

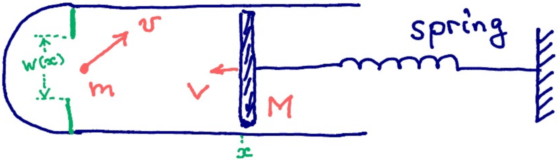
$$\frac{\langle v^2 \rangle}{\langle V^2 \rangle} = \frac{2m}{M} \ll 1$$

Hence typically the particle moves fast and the pistons moves slowly

ergodicity of the full system $\Longrightarrow$ time average of $\frac{MV^2}{2} \longrightarrow \frac{E}{4}$ (independently of its initial value)

ergodicity of the frozen billiard $\Longrightarrow$ $\frac{MV^2}{2}$ oscillates approximately periodically (in particular, it returns close to its initial value for $|t| < \frac{T}{\epsilon}$)

Example of non-ergodic fast dynamics (mushrooms)



$w(x)$ diameter of the gap left by the additional walls

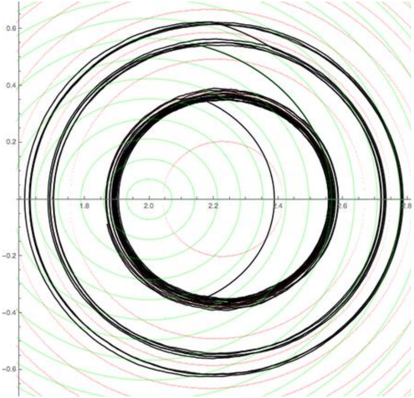
(x position of the piston)

on the plane $(x, \dot{x})$:

green lines: free motion of the piston

red lines: motion of the piston under the effective force obtained by averaging over the fast particle motion

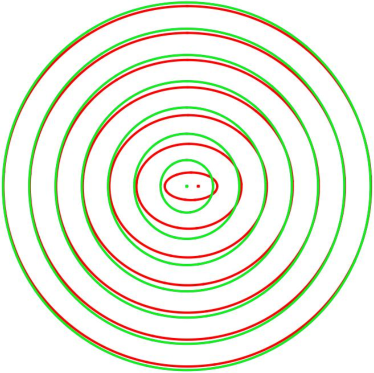
black line: a trajectory computed numerically



Limit model: random switching between two Hamiltonian vector fields

Random model

1. A point moves in the plane following a red line in the clockwise direction
2. At a random moment of time it changes to a green line
3. Then it follows the green line till it reaches the symmetric point, where it returns to the red line



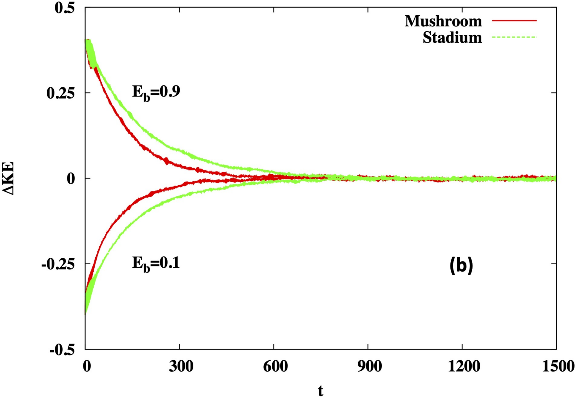
If it switches to green at (x_0, y_0), then it switches back to red at $(x_0, -y_0)$

The shapes of the lines and probability distribution for switching times are determined from the original deterministic system.

The model can be considered as random switching between two Hamiltonian vector fields

Numerics: evolution towards energy equipartition

Deviation of particles kinetic energy from $\frac{E}{4}$
(averaged over a large ensemble of initial conditions)



Numerical experiment:
Fix total energy E
Fix initial energy $E_p(0)$
Generate random initial conditions for the particle and the piston
Follow the dynamics for 1500 units of time
Plot the mean value of $E_p(t) - \frac{E}{4}$

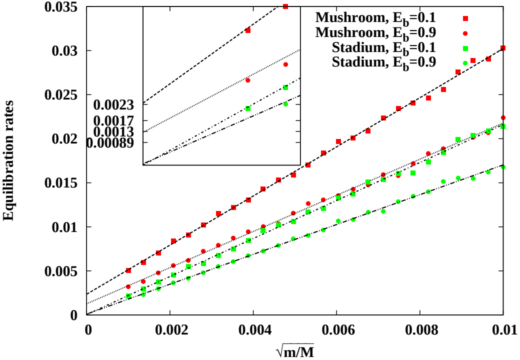
- convergence to the equipartition is exponential
- exponential rate depends on the initial energy $E_p(0)$
- equilibration is faster in the non-ergodic case
- exponential rate depends on the mass ratio $\frac{m}{M}$.

Numerics: extrapolation towards $\frac{m}{M} \to 0$

Numerical experiments confirm the closeness of the random model to the billiard with very small m.

Equilibration rates converge as $m \to 0$

- to zero in the ergodic case
- to a positive limit in the non-ergodic case
- the limit value is in agreement with the rate from the random model



Conclusions and open problems

Conclusions:

- Ergodicity of the fast dynamics can be an obstacle for ergodicity of the whole system
- Non-ergodicity of the fast dynamics can lead to faster energy equilibration
- The slow dynamics can be modelled by a curious random process

For a more detailed discussion see our paper in PNAS (2017).

Open problems:

- The majority of Hamiltonian systems are not ergodic. What is a right model for the slow dynamics in a more general case?
- What are the properties of the random process described in this talk?

Thank you!

The end