Gibbs measures of nonlinear Schrödinger equations as limits of many-body quantum states

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The nonlinear Schrödinger equation

Consider the spatial domain $\Lambda = \mathbb{T}^d$ for $d = 1, 2, 3$.

- Study the ***nonlinear Schrödinger equation (NLS)***.

$$\begin{cases} \mathrm{i}\partial_t \phi_t(x) = \big(-\Delta/2 + \kappa\big)\phi_t(x) + \int \mathrm{d}y\, w(x-y)\, |\phi_t(y)|^2\, \phi_t(x) \\ \phi_0(x) = \Phi(x) \in H^s(\Lambda)\,. \end{cases}$$

- Parameter: $\kappa > 0$.
- Interaction: $w : \Lambda \to \mathbb{R}$ is *positive* or $w = \delta$.
- Conserved energy

$$H(\phi) = \int \mathrm{d}x\, \bar{\phi}(x)(\kappa - \Delta/2)\phi(x) + \frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, |\phi(x)|^2\, w(x-y)\, |\phi(y)|^2\,.$$

Gibbs measures for the NLS

- The ***Gibbs measure*** $\mathrm{d}\mu$ associated with H is the probability measure on the space of fields $\phi : \Lambda \to \mathbb{C}$

$$\mu(\mathrm{d}\phi) := \frac{1}{Z} \mathrm{e}^{-H(\phi)} \, \mathrm{d}\phi \,, \qquad Z := \int \mathrm{e}^{-H(\phi)} \, \mathrm{d}\phi \,.$$

  $\mathrm{d}\phi =$ (formally-defined) Lebesgue measure.

- Formally, $\mathrm{d}\mu$ is invariant under the flow of the NLS:

$$(\mathrm{F}_t)_* \mathrm{d}\mu = \mathrm{d}\mu \,,$$

  where $\mathrm{F}_t :=$ flow map of NLS.

Gibbs measures for the NLS: known results

- **Rigorous construction of Gibbs measure:** CQFT literature in the 1970-s (Nelson, Glimm-Jaffe, Simon).
- **Proof of invariance:** Bourgain and Zhidkov (1990s).
  $\rightarrow$ Measure supported on low-regularity Sobolev spaces.
- **Application to PDE:** *Obtain low-regularity solutions of NLS μ-almost surely*.
  Recent advances: Bourgain-Bulut, Burq-Tzvetkov, Burq-Thomann-Tzvetkov, Cacciafesta- de Suzzoni, Deng-Nahmod-Yue, Fan-Ou-Staffilani-Wang, Genovese-Lucà-Valeri, Nahmod-Oh-Rey-Bellet-Staffilani, Nahmod-Rey-Bellet-Sheffield-Staffilani, Oh-Pocovnicu, Oh-Tzvetkov-Wang, Thomann-Tzvetkov, Tzvetkov, ...

Derivation of Gibbs measures: informal statement

NLS is a classical limit of many-body quantum theory.

- On $\mathfrak{H}^{(n)} \equiv L^2_{\mathrm{sym}}(\Lambda^n)$ we consider the n*-body Hamiltonian*

$$H^{(n)} := -\frac{1}{2}\sum_{i=1}^{n} \Delta_i + \frac{1}{n}\sum_{i,j=1}^{n} w(x_i - x_j)\,.$$

- Solve *n-body Schrödinger equation*

$$\mathrm{i}\partial_t \Psi_{n,t} = H^{(n)} \Psi_{n,t}\,.$$

  Obtain that, as $n \to \infty$

$$\Psi_{n,0} \sim \phi_0^{\otimes n} \quad \text{implies} \quad \Psi_{n,t} \sim \phi_t^{\otimes n}\,.$$

  (Hepp (1974), Ginibre-Velo (1979), Spohn (1980), Fröhlich-Tsai-Yau (1998), Fröhlich-Knowles-Pickl (2006), Erdős-Schlein-Yau (2006, 2007), Fröhlich-Graffi-Schwarz (2007), Fröhlich-Knowles-Schwarz (2009), T. Chen-Pavlović (2010), Pickl (2010), Ammari-Nier (2011), . . .).

- **Problem:** Obtain Gibbs measure $\mathrm{d}\mu$ as many-body quantum limit .

The Wiener measure and classical free field

- Let $H_0(\phi) := \int \mathrm{d}x\, (|\nabla\phi(x)|^2/2 + \kappa|\phi(x)|^2)$.
  Define the ***Wiener measure*** $\mathrm{d}\mu_0$

$$\mu_0(\mathrm{d}\phi) := \frac{1}{Z_0}\mathrm{e}^{-H_0(\phi)}\,\mathrm{d}\phi\,, \qquad Z_0 := \int \mathrm{e}^{-H_0(\phi)}\,\mathrm{d}\phi\,.$$

- For $\phi \in \operatorname{supp}\mathrm{d}\mu_0$,

$$\phi \equiv \phi^\omega \sim \sum_{k\in\mathbb{Z}^d} \frac{g_k(\omega)}{(|k|^2+\kappa)^{1/2}}\,\mathrm{e}^{2\pi i k\cdot x}\,,\ (g_k) = \text{i.i.d. complex Gaussians.}$$

  $\rightarrow$ ***Classical free field***.

- Series converges almost surely in $H^{1-\frac{d}{2}-\varepsilon}(\Lambda)$.

The classical system and Gibbs measures

- The *classical interaction* is

$$W := \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, |\phi^\omega(x)|^2\, w(x-y)\, |\phi^\omega(y)|^2\,.$$

- In $[0, +\infty)$ almost surely if $d = 1$ and $w \in L^\infty(\mathbb{T}^1)$ is *pointwise nonnegative*.
- In this case $\mathrm{d}\mu$ is a well-defined probability measure on $H^{1/2-\varepsilon}(\mathbb{T}^1)$ which satisfies

$$\mathrm{d}\mu \ll \mathrm{d}\mu_0\,.$$

- For $d = 2, 3$, *W is infinite almost surely* even if $w \in L^\infty(\mathbb{T}^d)$.

The classical system and Gibbs measures

- Perform a *renormalisation* in the form of ***Wick ordering***.

$$W^w \; := \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \left(|\phi^\omega(x)|^2 - \infty \right) w(x-y) \left(|\phi^\omega(y)|^2 - \infty \right).$$

- Rigorously defined as limit in $\bigcap_{m \geqslant 1} L^m(\mathrm{d}\mu_0)$ of truncations W_K .

$$\phi_K^\omega(x) \sim \sum_{|k| \leqslant K} \frac{g_k(\omega)}{\sqrt{|k|^2 + \kappa}} \, \mathrm{e}^{2\pi i k \cdot x},$$

$$\varrho_K(x) := \mathbb{E}_{\mu_0} |\phi_K^\omega(x)|^2 \sim \sum_{|k| \leqslant K} \frac{1}{|k|^2 + \kappa} \to \infty,$$

$$W_K := \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \left(|\phi_K^\omega(x)|^2 - \varrho_K \right) w(x-y) \left(|\phi_K^\omega(y)|^2 - \varrho_K \right).$$

- $W \equiv W^w \geqslant 0$ almost surely if $\hat{w}$ is *pointwise nonnegative*, i.e. w is of *positive type*: $\int \mathrm{d}x \, \mathrm{d}y \, f(x) w(x-y) f(y) = \sum_k \hat{w}(k) \, |\hat{f}(k)|^2$, for f real.

The classical system and Gibbs measures

- Classical Gibbs state $\rho(\cdot)$: Given $X \equiv X(\omega)$ a random variable, let

$$\rho(X) := \frac{\int X \, \mathrm{e}^{-W} \, \mathrm{d}\mu_0}{\int \mathrm{e}^{-W} \, \mathrm{d}\mu_0} = \int X \, \mathrm{d}\mu \,.$$

- On $\mathfrak{H}^{(p)} \equiv L^2_{\mathrm{sym}}(\Lambda^p)$ define the ***classical p-particle correlation function*** γ_p by its operator kernel

$$(\gamma_p)_{x_1,\ldots,x_p;y_1,\ldots,y_p} := \rho\big(\overline{\phi^\omega}(y_1) \cdots \overline{\phi^\omega}(y_p) \phi^\omega(x_1) \cdots \phi^\omega(x_p)\big) \,.$$

$\rightarrow$ μ is determined by $(\gamma_p)_p$.

The quantum problem

- Consider $d = 1$.
- Given $m > 0$ (mass of particles) and $\lambda > 0$ (coupling constant), we work with
$$H^{(n)} := \frac{1}{m} \sum_{i=1}^{n} \Big(-\frac{\Delta_i}{2} + \kappa\Big) + \frac{\lambda}{2} \sum_{i,j=1}^{n} w(x_i - x_j)\,.$$
- At inverse temperature $\beta \in (0, \infty)$, equilibrium of $H^{(n)}$ is governed by the *Canonical ensemble*
$$\frac{1}{Z_\beta^{(n)}} \mathrm{e}^{-\beta H^{(n)}}\,, \qquad Z_\beta^{(n)} := \operatorname{Tr} \mathrm{e}^{-\beta H^{(n)}}\,.$$
- Henceforth consider $\beta = 1$.
- We take $m = 1/\nu$ and $\lambda \sim \nu^2$ and analyse the regime $\nu \to 0$. This can be interpreted as a *mean-field/semiclassical limit*.

The quantum problem

- Work on the *Bosonic Fock space*

$$\mathcal{F} := \bigoplus_{n \in \mathbb{N}} \mathfrak{H}^{(n)}$$

with *quantum Hamiltonian*

$$H_\nu := \bigoplus_{n \in \mathbb{N}} H_\nu^{(n)}\,,$$

where

$$H_\nu^{(n)} := \nu \sum_{i=1}^{n} \Big(-\frac{\Delta_i}{2} + \kappa \Big) + \frac{\nu^2}{2} \sum_{i,j=1}^{n} w(x_i - x_j)\,.$$

- On $\mathcal{F}$ define the *grand canonical ensemble* by

$$\mathbb{P}_\nu := \frac{1}{Z_\nu} \bigoplus_{n \in \mathbb{N}} \mathrm{e}^{-H_\nu^{(n)}}\,, \qquad \mathrm{Tr}_{\mathcal{F}}\, \mathbb{P}_\nu = 1\,.$$

The quantum Gibbs state

- Work with *quantum fields (operator-valued distributions)* ϕ_ν, ϕ_ν^* on $\mathcal{F}$ that satisfy

$$[\phi_\nu(x), \phi_\nu^*(y)] = \nu\delta(x-y)\,, \quad [\phi_\nu(x), \phi_\nu(y)] = [\phi_\nu^*(x), \phi_\nu^*(y)] = 0\,.$$

Heuristic: $\phi_\nu \longleftrightarrow \phi^\omega$, $\phi_\nu^* \longleftrightarrow \overline{\phi^\omega}$.

- Quantum Gibbs state $\rho_\nu(\cdot)$: Given $\mathcal{A} \in \mathcal{L}(\mathcal{F})$ we define its expectation

$$\rho_\nu(\mathcal{A}) := \mathrm{Tr}_{\mathcal{F}}(\mathcal{A}\mathbb{P}_\nu)\,.$$

On $\mathfrak{H}^{(p)}$ define the ***quantum p-particle correlation function*** $\gamma_{\nu,p}$ by

$$(\gamma_{\nu,p})_{x_1,\ldots,x_p;y_1,\ldots,y_p} = \rho_\nu\big(\phi_\nu^*(y_1)\cdots\phi_\nu^*(y_p)\phi_\nu(x_1)\cdots\phi_\nu(x_p)\big)\,.$$

$\rightarrow \mathbb{P}_\nu$ is determined by $(\gamma_{\nu,p})_p$.

- One can write

$$(\gamma_{\nu,p})_{x_1,\ldots,x_p;y_1,\ldots,y_p} = \sum_{n \geqslant p} \frac{n!}{(n-p)!}\,\mathrm{Tr}_{p+1,\ldots,n}\left(\frac{\mathrm{e}^{-H_\nu^{(n)}}}{Z_\nu}\right).$$

Theorem 1: $w \in L^\infty$.

Theorem 1: Fröhlich, Knowles, Schlein, S. (CMP, 2017).

(i) Let $d = 1$ and $w \in L^\infty(\mathbb{T}^1)$ be pointwise nonnegative or $w = \delta$. Then for all $p \in \mathbb{N}$ we have

$$\gamma_{\nu,p} \xrightarrow{\mathrm{Tr}} \gamma_p \quad \text{as} \quad \nu \to 0\,.$$

The convergence is in the trace class. ($\|\mathcal{A}\|_{\mathrm{Tr}} := \mathrm{Tr}\,|\mathcal{A}|$).

(ii) Let $d = 2, 3$ and $w \in L^\infty(\mathbb{T}^d)$ be of positive type ($\hat{w} \geqslant 0$). The convergence holds in the Hilbert-Schmidt class *after a renormalisation procedure* (*Wick ordering*) and with a *slight modification of the grand canonical ensemble* $\mathbb{P}_\nu$ (needed for technical reasons).

Modification:

$$H_\nu = \underbrace{H_{\nu,0}}_{\text{Free part } (w=0)} + \underbrace{W_\nu}_{\text{Interaction}}\,.$$

$$\mathrm{e}^{-H_\nu} \mapsto \mathrm{e}^{-\eta H_{\nu,0}}\,\mathrm{e}^{-(1-2\eta)H_{\nu,0}-W_\nu}\,\mathrm{e}^{-\eta H_{\nu,0}}\,, \qquad \eta > 0\,.$$

Theorem 2: $w \in L^q$.

Theorem 2: S. (Preprint, 2019).

Let $d = 2, 3$ and $w \in L^q(\mathbb{T}^d)$ be of positive type, where

$$q \in \begin{cases} (1, \infty] \,, d = 2 \\ (3, \infty] \,, d = 3 \,. \end{cases}$$

With Wick renormalisation and modification of $\mathbb{P}_\nu$ as in Theorem 1, we have

$$\gamma_{\nu,p} \xrightarrow{\text{HS}} \gamma_p \quad \text{as} \quad \nu \to 0 \,.$$

$\to$ Optimal range of w for NLS: Bourgain (JMPA, 1997).

Theorem 3: L^1 interactions in two dimensions.

Theorem 3: S. (Preprint, 2019).

Let $d = 2$ and $w \in L^1(\mathbb{T}^d)$ satisfy the following assumptions.

- w is of positive type.
- w is pointwise nonnegative.
- There exist $\varepsilon > 0$ and $C > 0$ such that

$$\widehat{w}(k) \leqslant \frac{C}{(1+|k|)^{\varepsilon}}$$

for all $k \in \mathbb{Z}^2$.

With setup as in Theorem 1, we have

$$\gamma_{\nu,p} \xrightarrow{\text{HS}} \gamma_p \quad \text{as} \quad \nu \to 0\,.$$

$\rightarrow$ Classical variant of this endpoint case: Bourgain (JMPA, 1997).

Unmodified grand canonical ensemble 1

Theorem 4: Fröhlich, Knowles, Schlein, S. (Preprint, 2020).

Suppose that the interaction w is *continuous* and of *positive type*. Then for all $p \in \mathbb{N}$ we have

$$\gamma_{\nu,p} \xrightarrow{L^r} \gamma_p \quad \text{as} \quad \nu \to 0\,,$$

where

$$r \in \begin{cases} [1,\infty]\,, & d=1 \\ [1,\infty)\,, & d=2 \\ [1,3)\,, & d=3\,. \end{cases}$$

is optimal. We apply Wick ordering when $d = 2, 3$.
The result holds with an *unmodified grand canonical ensemble* $\mathbb{P}_\nu$.

Unmodified grand canonical ensemble 2

Theorem 4 can be deduced from a stronger result.
We Wick order $\gamma_{\nu,p}$ and γ_p to obtain $\widehat{\gamma}_{\nu,p}$ and $\widehat{\gamma}_p$.
Example: We have

$$\widehat{\gamma}_1 = \gamma_1 - \gamma_1^0$$

and

$$(\widehat{\gamma}_2)_{x_1,x_2;\tilde{x}_1,\tilde{x}_2} = (\gamma_2)_{x_1,x_2;\tilde{x}_1,\tilde{x}_2} - (\gamma_1)_{x_1;\tilde{x}_1}\,(\gamma_1^0)_{x_2;\tilde{x}_2} - (\gamma_1)_{x_1;\tilde{x}_2}\,(\gamma_1^0)_{x_2;\tilde{x}_1} - (\gamma_1)_{x_2;\tilde{x}_1}\,(\gamma_1^0)_{x_1;\tilde{x}_2} - (\gamma_1)_{x_2;\tilde{x}_2}\,(\gamma_1^0)_{x_1;\tilde{x}_1} + (\gamma_2^0)_{x_1,x_2;\tilde{x}_1,\tilde{x}_2}\,.$$

Theorem 5: Fröhlich, Knowles, Schlein, S. (Preprint, 2020).

For all $p \in \mathbb{N}$ we have

$$\widehat{\gamma}_{\nu,p} \xrightarrow{C} \widehat{\gamma}_p \quad \text{as} \quad \nu \to 0\,,$$

where $\xrightarrow{C}$ denotes convergence in the space of continuous functions w.r.t. $\|\cdot\|_{L^\infty}$.

Related results

- $1D$ results: previously shown using variational techniques by Lewin, Nam, Rougerie (J. Éc. Polytech. Math., 2015).
  Higher dimensions: non local, non translation-invariant interactions.
- Lewin, Nam, Rougerie (JMP, 2018) : $1D$ non-periodic problem with subharmonic trapping.
- Lewin, Nam, Rougerie (preprint 2018) : $2D$ problem with translation-invariant interaction without modified Gibbs state.
- Lewin, Nam, Rougerie (preprint 2020) : Extension to $3D$.
- Fröhlich, Knowles, Schlein, S. (AIM 2019): time-dependent problem in $1D$. $\rightarrow$ *Corresponds to the invariance of the measure*.

The $\nu \to 0$ limit in the free case

Examine the limit $\nu \to 0$ in the *free case* $w = 0$.

- Define the *rescaled particle number operator* by

$$\mathcal{N}_\nu := \nu \bigoplus_{n \in \mathbb{N}} n \mathrm{I}_{\mathfrak{H}^{(n)}} = \int \mathrm{d}x \, \phi_\nu^*(x) \, \phi_\nu(x) \, .$$

- Compare with

$$\mathcal{N} := \int \mathrm{d}x \, |\phi^\omega(x)|^2 \, .$$

- We have

$$\rho_\nu(\mathcal{N}_\nu) \sim \sum_{k \in \mathbb{Z}^d} \frac{\nu}{\mathrm{e}^{\nu(|k|^2 + \kappa)} - 1} \sim \begin{cases} 1 & \text{if } d = 1 \\ \log \nu^{-1} & \text{if } d = 2 \\ \nu^{-1/2} & \text{if } d = 3 \, . \end{cases}$$

  $\rho_\nu(\cdot)$ has a *natural cut-off* for $|k| \geqslant \nu^{-1/2}$.

$\to$ *Need to renormalise when* $d = 2, 3$.

Methods of proof

1. Theorems $1-3$: apply a *perturbative expansion* in the interaction and a resummation of the resulting series.

$$H_\nu = \underbrace{H_{\nu,0}}_{\text{Free part } (w=0)} + \underbrace{W_\nu}_{\text{Interaction}} .$$

2. Theorems $4-5$: Use a *functional integral* formulation.
$\rightarrow$ Represent the field theory as a gas of *interacting Brownian loops and paths* : Ginibre (1965), Symanzik (1968).

Method 1: Perturbative expansion

- **Example**: Consider the

Classical (relative) partition function $\zeta(z) := \int \mathrm{e}^{-zW} \, \mathrm{d}\mu_0$

Quantum (relative) partition function $\zeta_\nu(z) = \dfrac{\mathrm{Tr}_{\mathcal{F}}\left(\mathrm{e}^{-H_{\nu,0}-zW_\nu}\right)}{\mathrm{Tr}_{\mathcal{F}}(\mathrm{e}^{-H_{\nu,0}})}$.

- $\zeta(z)$ and $\zeta_\nu(z)$ are analytic in $\mathrm{Re}\, z > 0$.
- Our goal is to prove that

$$\lim_{\nu \to 0} \zeta_\nu(z) = \zeta(z) \quad \text{for } \mathrm{Re}\, z > 0 .$$

- **Problem:** The series expansions

$$\zeta(z) = \sum_{m=0}^{\infty} a_m z^m , \quad \zeta_\nu(z) = \sum_{m=0}^{\infty} a_{\nu,m} z^m$$

have *radius of convergence zero*.

Borel resummation

- Given a formal power series

$$\mathcal{A}(z) = \sum_{m \geqslant 0} \alpha_m z^m$$

its *Borel transform* is

$$\mathcal{B}(z) := \sum_{m \geqslant 0} \frac{\alpha_m}{m!} z^m .$$

Formally we have $\mathcal{A}(z) = \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-t} \, \mathcal{B}(tz)$.

Example (1)

If $\mathcal{A}(z) = \sum_{m \geqslant 0} z^m$ then $\mathcal{B}(z) = \sum_{m \geqslant 0} \frac{z^m}{m!} = e^z$ and we have

$$\int_0^\infty \mathrm{d}t \, \mathrm{e}^{-t} \, \mathrm{e}^{tz} = \frac{1}{1-z} , \text{ for } \operatorname{Re} z < 1 .$$

Example (2)

$f(z) = \int \mathrm{e}^{-zx^4} \, \mathrm{e}^{-x^2/2} \, \mathrm{d}x$: analytic on $\operatorname{Re} z > 0$, radius of convergence is zero.

Borel resummation

Write

$$\zeta(z) = \sum_{m=0}^{M-1} a_m z^m + R_M(z)\,, \quad \zeta_\nu(z) = \sum_{m=0}^{M-1} a_{\nu,m} z^m + R_{\nu,M}(z)\,.$$

By Sokal (1980), we should prove the following.

(i) The *explicit terms* satisfy

$$|a_m| + |a_{\nu,m}| \leqslant C^m m!\,.$$

The *remainder terms* satisfy

$$|R_M(z)| + |R_{\nu,M}(z)| \leqslant C^M M! |z|^M\,.$$

$\rightarrow$ In $2D, 3D$, bound on $R_{\nu,M}$ requires modification of $\mathbb{P}_\nu$.

(ii) The quantum coefficients converge to the classical coefficients, i.e.

$$\lim_{\nu \to 0} a_{\nu,m} = a_m\,.$$

Optimality of q $(w \in L^q)$

- Recall the classical expansion

$$\zeta(z) \sim \sum_{m=0}^{\infty} a_m z^m$$

- We can compute

$$a_1 \sim \int \mathrm{d}x \int \mathrm{d}y \, w(x-y) \, (G_{x,y})^2 \, .$$

where $G = (-\Delta/2 + \kappa)^{-1}$.

- We have $G \in L^r(\mathbb{T}^d \times \mathbb{T}^d)$ where

$$r \in \begin{cases} [1, \infty) \, , & d = 2 \\ [1, 3) \, , & d = 3 \, . \end{cases}$$

$\rightarrow$ Obtain optimal range of q by duality.

Method 2: Functional integral formulation

- Consider $d = 1$.
- Classical field $\phi : \Lambda \to \mathbb{C}$, measure $\mathrm{d}\phi := \prod_{x \in \Lambda} \phi(x)$.
- Define

$$
\begin{aligned}
S^0(\phi) &:= \int_\Lambda \mathrm{d}x\, \bar{\phi}(x)(\kappa - \Delta/2)\phi(x) \\
W(\phi) &:= \frac{1}{2} \int_{\Lambda^2} \mathrm{d}x\, \mathrm{d}y\, |\phi(x)|^2\, w(x-y)\, |\phi(y)|^2 \\
S(\phi) &:= S^0(\phi) + W(\phi)\,.
\end{aligned}
$$

Classical (relative) partition function:

$$
\zeta = \frac{\int \mathrm{d}\phi\, \mathrm{e}^{-S(\phi)}}{\int \mathrm{d}\phi\, \mathrm{e}^{-S^0(\phi)}}\,.
$$

Functional integral (formal setup)

- Quantum field $\Phi : [0,\nu] \times \Lambda \to \mathbb{C}$, measure $\mathrm{D}\Phi := \prod_{\tau \in [0,\nu]} \prod_{x \in \Lambda} \Phi(\tau, x)$.
- Define

$$\mathbf{S^0}(\Phi) := \int_0^\nu \mathrm{d}\tau \int_\Lambda \mathrm{d}x\, \bar{\Phi}(\tau,x)(\partial_\tau + \kappa - \Delta/2)\Phi(\tau,x)$$
$$\mathbf{W}(\Phi) := \frac{1}{2}\int_0^\nu \mathrm{d}\tau \int_{\Lambda^2} \mathrm{d}x\,\mathrm{d}y\, |\Phi(\tau,x)|^2\, w(x-y)\, |\Phi(\tau,y)|^2$$
$$\mathbf{S}(\Phi) := \mathbf{S^0}(\Phi) + \mathbf{W}(\Phi)\,.$$

- Quantum (relative) partition function $\mathcal{Z}_\nu = \frac{\int \mathrm{D}\Phi\, \mathrm{e}^{-\mathbf{S}(\Phi)}}{\int \mathrm{D}\Phi\, \mathrm{e}^{-\mathbf{S^0}(\Phi)}}$ (formally).
- Rescale for $t \in [0,1]$ as $\Phi'(t,x) := \sqrt{\nu}\Phi(\nu t, x)$.

$$\mathbf{S}(\Phi) = \int_0^1 \mathrm{d}t \int_\Lambda \mathrm{d}x\, \bar{\Phi}'(t,x)\big(\partial_t/\nu + \kappa - \Delta/2\big)\Phi'(t,x) + \frac{1}{2}\int_0^1 \mathrm{d}t \int_{\Lambda^2} \mathrm{d}x\,\mathrm{d}y\, |\Phi'(t,x)|^2\, w(x-y)\, |\Phi'(t,y)|^2\,.$$

- Formally deduce $\mathcal{Z}_\nu \to \zeta$ by stationary phase.

Functional integral (formal setup)

- Field theory $\mathrm{e}^{-\mathbf{S}(\Phi)}\,\mathrm{D}\Phi$: previously studied by Chen, Fröhlich, Seifert (1994), Moshe, Zinn-Justin (2003).
- The free theory $\mathrm{e}^{-\mathbf{S^0}(\Phi)}\,\mathrm{D}\Phi$, with

$$\mathbf{S^0}(\Phi) = \int_0^\nu \mathrm{d}\tau \int_\Lambda \mathrm{d}x\,\bar{\Phi}(\tau,x)(\partial_\tau + \kappa - \Delta/2)\Phi(\tau,x)$$

  does not yield a well-defined Gaussian measure: Cameron (1962).
- Instead, one works with correlation functions, which can be rigorously interpreted.

Hubbard-Stratonovich transformation

- Let $\mathcal{C} > 0$ be an $n \times n$ matrix. The Gaussian probability measure on $\mathbb{R}^n$ with covariance $\mathcal{C}$ is

$$\mu_{\mathcal{C}}(\mathrm{d}u) := \frac{1}{\sqrt{(2\pi)^n \det \mathcal{C}}} \, \mathrm{e}^{-\frac{1}{2}\langle u, \mathcal{C}^{-1} u\rangle} \, \mathrm{d}u \,.$$

- **Wick's theorem:** for any $f \in \mathbb{R}^n$ we have

$$\int \mu_{\mathcal{C}}(\mathrm{d}u) \, \mathrm{e}^{\mathrm{i}\langle f, u\rangle} = \mathrm{e}^{-\frac{1}{2}\langle f, \mathcal{C} f\rangle} \,.$$

- **Hubbard-Stratonovich transformation:** for a real Gaussian measure $\mu_{\mathcal{C}}$ (not necessarily finite dimensional) with covariance $\mathcal{C}$ we have

$$\int \mu_{\mathcal{C}}(\mathrm{d}\sigma) \, \mathrm{e}^{\mathrm{i}\langle f, \sigma\rangle} = e^{-\frac{1}{2}\langle f, \mathcal{C} f\rangle} \,.$$

(In general formal if f and σ are both rough!)

Functional integral + HS transformation

- Consider $\sigma : [0,\nu] \times \Lambda \to \mathbb{R}$ centred with law $\mu_{\mathcal{C}}$ and covariance

$$\int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \sigma(\tau,x)\, \sigma(\tilde{\tau},\tilde{x}) = \nu\, \delta(\tau-\tilde{\tau})\, w(x-\tilde{x}) \equiv \mathcal{C}^{\tau,\tilde{\tau}}_{x,\tilde{x}}\,.$$

- (Formally) use HS with $f = |\Phi|^2$ and let $K(u) := \partial_\tau - \Delta/2 - u$

$$\mathcal{Z}'_\nu := \frac{\int \mathrm{D}\Phi\, \mathrm{e}^{-\mathbf{S}(\Phi)}}{\int \mathrm{D}\Phi\, \mathrm{e}^{-\mathbf{S^0}(\Phi)}} = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \frac{\int \mathrm{D}\Phi\ \exp\big(-\langle \Phi, \overbrace{(\partial_\tau - \Delta/2 + \kappa - \mathrm{i}\sigma)}^{K(-\kappa+\mathrm{i}\sigma)}\Phi\rangle\big)}{\int \mathrm{D}\Phi\ \exp\big(-\langle \Phi, (\partial_\tau - \Delta/2 + \kappa)\Phi\rangle\big)}\,.$$

By Gaussian integration

$$\mathcal{Z}'_\nu = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \frac{\det K(-\kappa+\mathrm{i}\sigma)^{-1}}{\det K(-\kappa)^{-1}} = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \mathrm{e}^{F_1(\sigma)}\,,$$

$$F_1(\sigma) := \int_0^\infty \mathrm{d}t\ \mathrm{Tr}\bigg(\frac{1}{t+K(-\kappa+\mathrm{i}\sigma)} - \frac{1}{t+K(-\kappa)}\bigg)\,.$$

We used $\det(A) = \exp\big(\mathrm{Tr}\log A\big), \log a - \log b = -\int_0^\infty \mathrm{d}t\big(\frac{1}{t+a} - \frac{1}{t+b}\big).$

Space-time representation

- **Goal:** Find $K(u)^{-1}$ for $K(u) = \partial_\tau - \Delta/2 - u$.
- **Toy example:** $\mathcal{K} = \partial_\tau + \kappa$ on $L^2([0,\nu])$.

$$(\mathcal{K}^{-1})^{\tau,\tilde{\tau}} = \sum_{r \in \nu\mathbb{N}} \mathbf{1}_{\tau+r>\tilde{\tau}} \, \mathrm{e}^{-\kappa(\tau-\tilde{\tau}+r)} \,.$$

- We have

$$(K(u)^{-1})^{\tau,\tilde{\tau}}_{x,\tilde{x}} = \sum_{r \in \nu\mathbb{N}} \mathbf{1}_{\tau+r>\tilde{\tau}} \, W^{\tau+r,\tilde{\tau}}_{x,\tilde{x}}(u) \,,$$

where for $[t]_\nu := (t \bmod \nu) \in [0,\nu)$, $(W^{\tau,\tilde{\tau}})_{\tilde{\tau}\leqslant\tau}$ solves

$$\partial_\tau W^{\tau,\tilde{\tau}}(u) = \left(\frac{1}{2}\Delta + u([\tau]_\nu)\right) W^{\tau,\tilde{\tau}}(u) \,, \qquad W^{\tau,\tau}(u) = 1 \,.$$

Feynman-Kac formula: the kernel of $W^{\tau,\tilde{\tau}}$ is given by

$$W^{\tau,\tilde{\tau}}_{x,\tilde{x}}(u) = \int \mathbb{W}^{\tau,\tilde{\tau}}_{x,\tilde{x}}(\mathrm{d}\omega) \, \mathrm{e}^{\int_{\tilde{\tau}}^{\tau} \mathrm{d}t \, u([t]_\nu, \omega(t))} \,.$$

Conclusion: $\mathcal{Z}'_\nu = \int \mathrm{e}^{-\mathbf{S}(\Phi)} \, \mathrm{D}\Phi$ is a rigorous expression in terms of Brownian loops; similarly for correlation functions.
$\to$ Ginibre representation.

Functional integral representation

- By the Feynman-Kac formula and Hubbard-Stratonovich transformation and working backwards, we formally get

$$\mathcal{Z}_\nu = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \mathrm{e}^{F_1(\sigma)}$$

  (the true quantum partition function).
- In practice, always regularise $\mathcal{C}$.
- Replace $\mathcal{C} \mapsto \mathcal{C}_\eta$ for $\eta > 0$, such that under the law of $\mu_{\mathcal{C}_\eta}$, σ is almost surely smooth.

$$\int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \sigma(\tau,x)\, \sigma(\tilde{\tau},\tilde{x}) = \nu\, \delta_{\eta,\nu}(\tau-\tilde{\tau})\, w_\eta(x-\tilde{x}) =: (\mathcal{C}_\eta)^{\tau,\tilde{\tau}}_{x,\tilde{x}}\,.$$

- We have

$$\mathcal{Z}_\nu = \lim_{\eta\to 0} \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \mathrm{e}^{F_1(\sigma)}\,.$$

Wick ordering

- Wick order for $d = 2, 3$.

$$\mathcal{Z}_\nu = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \mathrm{e}^{F_2(\sigma)}$$

where

$$F_2(\sigma) := \int_0^\infty \mathrm{d}t \; \mathrm{Tr}\biggl(\frac{1}{t + K(-\kappa + \mathrm{i}\sigma)} - \frac{1}{t + K(-\kappa)} - \frac{1}{t + K(-\kappa)}\, \mathrm{i}\sigma\, \frac{1}{t + K(-\kappa)} \biggr).$$

- We subtract the first order term in the resolvent expansion.

Functional integral representation: classical setting

- **Classical setting:** derive a similar representation, after Symanzik (1968).
- μ_w: real Gaussian measure with mean zero and covariance

$$\int \mu_w(\mathrm{d}\xi)\, \xi(x)\, \xi(\tilde{x}) = w(x - \tilde{x})\,.$$

- We have

$$\zeta = \int \mu_w(\mathrm{d}\xi)\, \mathrm{e}^{f_2(\xi)}\,,$$

where

$$f_2(\xi) := \int_0^\infty \mathrm{d}t\ \mathrm{Tr}\biggl(\frac{1}{t - \Delta/2 + \kappa - \mathrm{i}\xi} - \frac{1}{t - \Delta/2 + \kappa} - \frac{1}{t - \Delta/2 + \kappa}\,\mathrm{i}\xi\,\frac{1}{t - \Delta/2 + \kappa}\biggr)\,.$$

Conclusion of the proof

- Use the functional integral representations

$$\mathcal{Z}_\nu = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \mathrm{e}^{F_2(\sigma)}\,, \qquad \zeta = \int \mu_w(\mathrm{d}\xi)\, \mathrm{e}^{f_2(\xi)}$$

  to obtain $\mathcal{Z}_\nu \to \zeta$ as $\nu \to 0$.
- **Fact:** If σ has law $\mu_{\mathcal{C}}$, then

$$\langle \sigma \rangle := \frac{1}{\nu} \int_0^\nu \mathrm{d}\tau\, \sigma(\tau, x)$$

  has law μ_w.
- Show that

$$\lim_{\nu \to 0} \int \mu_{\mathcal{C}}(\mathrm{d}\sigma) \left[\mathrm{e}^{F_2(\sigma)} - \mathrm{e}^{f_2(\langle \sigma \rangle)} \right] = 0\,.$$

- Conclude the proof by analysing Riemann sums and using continuity properties of Brownian paths.

Thank you for your attention!