How many stable equilibria will a large complex system have?

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based on collaborative work with Yan Fyodorov (PNAS 2016)
Gerárd Ben Arous and Yan Fyodorov (in preparation)
Jacek Grela (in preparation)

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Will a large complex system be stable? (Robert May, 1972)

Context: diversity vs. stability debate in ecology, 1960s

May's local stability analysis and 'minimal' linear model:

- Start with system of nonlinear ODEs $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, $\mathbf{y} \in \mathbf{R}^N$, $N \gg 1$.
- Use linear approximation (this will do for generic non-linear systems). Assume equilibrium at $\mathbf{y} = \mathbf{0}$. Then have a linear model:

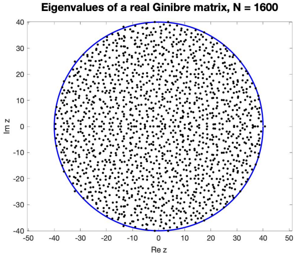
  $$\dot{\mathbf{y}} = -\mu \mathbf{y} + J\mathbf{y}, \qquad \mu > 0.$$

  Parameter μ sets the relaxation time scale.

  In an ecological context, $y_j(t)$ is the variation in popn dens of species j, and J_{jk} measures per capita effect of species k on j, hence J is asymmetric.
- Have local stability at $\mathbf{y} = 0$ iff all EVs of $J_{jk} - \mu\delta_{jk}$ have negative real parts ... For large complex systems, one can't hope to work out all J_{jk} in sufficient detail ... Robert May assumed random interaction instead:

  $$\langle J_{jk} \rangle = 0 \qquad \langle J_{jk}^2 \rangle = \alpha^2.$$

Circular Law and May-Wigner Instability Transition



EVs of J are uniformly distributed over the disk $|z| \leq \alpha\sqrt{N}$ in the limit $N \gg 1$

Ginibre 1965 (complex Gaussian), Edelman 1998 (real Ginibre), Tau & Vu 2010 (iid, Circular Law).

Law of fluctuations of the largest real part about its mean value $\alpha\sqrt{N}$ is still an open problem, however large deviations can be worked out.

May 1972: For large N the largest real part of EV of J is typically $\alpha\sqrt{N}$ and $-\mu\delta_{jk} + J_{jk}$ is almost certainly stable if $\frac{\mu}{\alpha\sqrt{N}} > 1$ and unstable if $\frac{\mu}{\alpha\sqrt{N}} < 1$.

In May's words: "The central feature of the above results for large systems is the very sharp transition from stable to unstable behaviour as the complexity ... exceeds a critical value ('May-Wigner theorem'). "

Local stability analysis v global picture

Linearisation give access to local behaviour, and the May-Wigner theorem simply implies breakdown of linear approximation for large complex systems as complexity exceeds a critical value.

In other words, the linear framework, despite being so popular, gives no answer to the question about what is happening to the *original* system when it loses stability.

Is there a signature of the May-Wigner instability transition on the global scale?

Seems natural to study statistics of numbers of equilibria (EQA) of large complex systems in the first instance. E.g., how many stable EQA will a large complex system have?

Nonlinear systems: statistics of equilibria

Consider a system of nonlinear ODEs

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^N, \quad N \gg 1.$$

$\mathbf{F}(\mathbf{x})$ is a smooth random vector field. To find stationary points (equilibria), have to solve $\mathbf{F}(\mathbf{x}) = 0$. Hardly possible. Also, the no of EQA and their positions change from one realisation to another. Statistical approach?

Q: Pick an EQM $\mathbf{x}_*$ at random. What is the prob for it to be stable?

Stable means locally stable, $x_{max}(\mathrm{Jac}(\mathbf{x}_*)) < 0$, where $\mathrm{Jac} = (\partial F_i / \partial x_j)_{ij}$ and $x_{max}(J)$ is the maximal real part of egv of J.

A: $p_{st} = \langle \mathcal{N}_{st} / \mathcal{N}_{tot} \rangle$, where $\mathcal{N}_{st}$ is the no of stable EQA, $\mathcal{N}_{tot}$ is the total no of EQA, and $\langle \ldots \rangle$ is the average over the realisations of $\mathbf{F}(\mathbf{x})$.

Calculating p_{st} is difficult task. Instead, use annealed approximation:

$$p_{st}^{a} = \langle \mathcal{N}_{st} \rangle / \langle \mathcal{N}_{tot} \rangle$$

‘Minimal’ nonlinear model for large complex systems (Yan Fyodorov-BK 2016)

$$\dot{\mathbf{x}} = -\mu \mathbf{x} + \mathbf{f}(\mathbf{x}), \quad \mu > 0.$$

Have stability if no interaction. Random interaction $\mathbf{f}(\mathbf{x})$:

$$f_i(\mathbf{x}) = -\frac{\partial V}{\partial x_i} + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\partial A_{ij}}{\partial x_j}, \quad A_{ij}(\mathbf{x}) = -A_{ji}(\mathbf{x}) \; \forall i, j.$$

This is a fairly general class, recall Helmholtz's $\mathbf{f} = \nabla V + \nabla \times \mathbf{A}$ for $N = 3$.

Our $\mathbf{f}$ has ‘gradient’ (irrotational) and ‘solenoidal’ parts.

Working assumptions: V and A_{ij} are independent, homogeneous, isotropic Gaussian fields with zero mean and covariances

$$\begin{aligned} \langle V(\mathbf{x}) V(\mathbf{y}) \rangle &= v^2 G_V\left(|\mathbf{x}-\mathbf{y}|^2\right), \\ \langle A_{ij}(\mathbf{x}) A_{nm}(\mathbf{y}) \rangle &= a^2 G_A\left(|\mathbf{x}-\mathbf{y}|^2\right)(\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}) \end{aligned}$$

Also assume finite 3rd moments and normalise $d^2 G_{V,A}(s)/ds^2\big|_{s=0} = 1$

‘Minimal’ nonlinear model $\dot{\mathbf{x}} = -\mu\mathbf{x} + \mathbf{f}(\mathbf{x})$

This model has two parameters (fewest possible):

$m = \frac{\mu}{\alpha\sqrt{N}}$, $\alpha = 2\sqrt{v^2 + a^2}$.

relaxation strength relative to interaction, similar to May's.

$\tau = \frac{v^2}{v^2 + a^2}$.

balance between longitudinal and transverse components of $\mathbf{f}$.

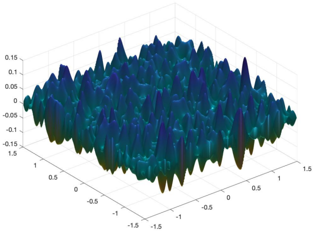
Pure gradient flow ($\tau = 1$) visualised as gradient descent on a random surface.

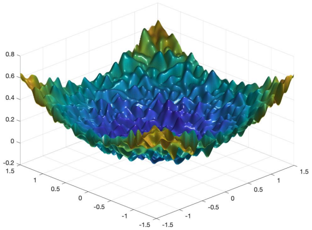
In this case have $\dot{\mathbf{x}} = -\nabla L(\mathbf{x})$ where $L(\mathbf{x}) = \frac{1}{2}\mu|\mathbf{x}|^2 - V(\mathbf{x})$ (Lyapunov fnc).

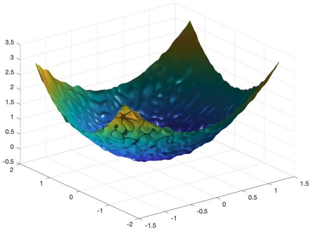
- Dynamics: $\mathbf{x}(t)$ moves in the direction steepest descent
- EQM are critical points, stable EQM are local minima on surface $h = L(\mathbf{x})$.
- $\frac{1}{2}\mu|\mathbf{x}|^2$ confining potential, deep well on the surface if μ is large;
  switching on random potential $V(\mathbf{x})$ results in multitude of shallow wells.

Studied in the context of random energy landscapes by Fyodorov from 2004.

Random surfaces







‘Minimal’ model $\dot{\mathbf{x}} = -\mu\mathbf{x} + \mathbf{f}(\mathbf{x})$: counting equilibria (EQA) via Kac-Rice

(i) EQA are roots of $-\mu\mathbf{x} + \mathbf{f}(\mathbf{x})$. Total no of EQA is $\mathcal{N}_{tot}$. Then

$$\langle \mathcal{N}_{tot} \rangle = \int \langle \delta\left(\mathbf{F}(\mathbf{x})\right) \left|\det\left(\partial F_i / \partial x_j\right)\right| \rangle \, d\mathbf{x} \qquad \text{(Kac-Rice).}$$

homogeneity + Gaussianity $\implies$ $\mathbf{F}(\mathbf{x})$ and $(\partial F_i / \partial x_j)$ are independent

$$\therefore \quad \langle \mathcal{N}_{tot} \rangle = \frac{1}{\mu^N} \left\langle \left| \det\left(\frac{\partial F_i}{\partial x_j} \right) \right| \right\rangle .$$

To leading order in N,
$$\left(\frac{\partial F_i}{\partial x_j} \right) \stackrel{d}{=} \alpha\sqrt{N}(-\xi I + X),$$

where scalar ξ and matrix X are independent, $\xi \sim N(m, \tau/N)$ and

$$P(X) \propto \exp\left[-\frac{N}{2(1-\tau^2)} \left(\operatorname{Tr} X X^T - \tau \operatorname{Tr} X^2 \right) \right] \qquad \text{(elliptic Ginibre)}$$

$$\therefore \quad \boxed{\langle \mathcal{N}_{tot} \rangle = \frac{1}{m^N} \int_{-\infty}^{\infty} \langle \left|\det\left[X - xI\right]\right| \rangle_X \, \frac{e^{-\frac{N(x-m)^2}{2\tau}} dx}{\sqrt{2\pi\tau/N}},}$$

(ii) analytic problem: find the average of the abs value of the char. polynomial.

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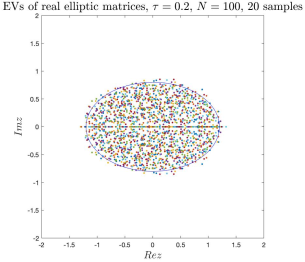
To leading order in N, $\left(\frac{\partial F_i}{\partial x_j}\right) \overset{d}{=}$



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‘Minimal’ model $\dot{\mathbf{x}} = -\mu\mathbf{x} + \mathbf{f}(\mathbf{x})$: Counting total no of equilibria $\mathcal{N}_{tot}$

Thm [Yan Fydorov and BK 2016] Assume $\tau < 1$ (and N even, technical).

Then, to leading order in the limit $N \gg 1$,

$$\langle \mathcal{N}_{tot} \rangle = \begin{cases} 1 & \text{if } m > 1; \\ \sqrt{\frac{2(1+\tau)}{1-\tau}}\, e^{N\,\Sigma_{tot}(m)} & \text{if } 0 < m < 1. \end{cases}$$

where the complexity exponent $\Sigma_{tot}(m) = \frac{m^2-1}{2} - \ln m > 0 \quad (0 < m < 1)$.

Note $\Sigma_{tot}(m) \sim (1-m)^2$ as $m \uparrow 1$. Hence, the width of the transition region is prop to $N^{-1/2}$. The *crossover profile of* $\langle \mathcal{N}_{tot} \rangle$ in this region *can be obtained in closed form*.

Note singularity in the-exponential term. The $\tau = 1$ limit can be accessed via scaling $\tau = 1 - \frac{u^2}{\sqrt{N}}$ and $\langle \mathcal{N}_{tot} \rangle$ *can be obtained in closed form* in this limit.

Key element of proof: this is based on Edelman, Kostlan & Shub (1994)

Consider $(N+1)\times(N+1)$ matrices X_{N+1}.

If x is a real eigenvalue of X_{N+1} then $X_{N+1} = Q\begin{pmatrix} x & \mathbf{w} \\ 0 & X_N \end{pmatrix} Q^T$.

The Jacobian of changing from X_{N+1} to $X_N, Q, x, \mathbf{w}$ is $|\det(xI_N - X_N)|$.

Note: $\mathrm{Tr}\, X_{N+1}X_{N+1}^T = x^2 + \mathbf{w}^T\mathbf{w} + \mathrm{Tr}\, X_N X_N^T$ and $\mathrm{Tr}\, X_{N+1}^2 = x^2 + \mathrm{Tr}\, X_N^2$.

Therefore, if X_{N+1} is elliptic, then X_N is so too. This implies

$$\boxed{\rho_{N+1}^{(r)}(x) = \frac{(N-2)!!}{(N-1)!}\,\frac{e^{-\frac{x^2}{2(1+\tau)}}}{2\sqrt{1+\tau}}\,\langle|\det(xI - X)|\rangle_{X_N}}$$

where $\rho_{N+1}^{(r)}(x)$ is the mean density of real eigenvalues in the real elliptic ensemble X_{N+1} of $(N+1)\times(N+1)$ matrices , and the average on the right is over the real elliptic ensemble X_N of $N\times N$.

How many equilibria are stable?

Averaged number of stable equilibria $\langle \mathcal{N}_{st} \rangle$ via Rice-Kac:

$$\langle \mathcal{N}_{st} \rangle = \frac{1}{m^N} \int_{-\infty}^{\infty} \langle \Theta(x - x_{max}) | \det (X - xI) | \rangle_X \frac{e^{-\frac{N(x-m)^2}{2\tau}} dx}{\sqrt{2\pi\tau/N}},$$

where x_{max} is the max real part of EVs of X and $\Theta(x - x_{max})$ is the indicator function of the event $x > x_{max}$ (i.e., the linearised system is stable).

Pure gradient flow $\tau = 1$ is special: all EVs are real, have exact relation

$$\frac{d}{dx} \langle \Theta(x - x_{max}) \rangle_{X_{N+1}} = c_N \, e^{-\frac{x^2}{4}} \, \langle \Theta(x - x_{max}) \, |\det (X - xI)| \rangle_{X_N}$$

The LHS is known in the limit $N \gg 1$ (Tracy-Widom 1994). This helps!

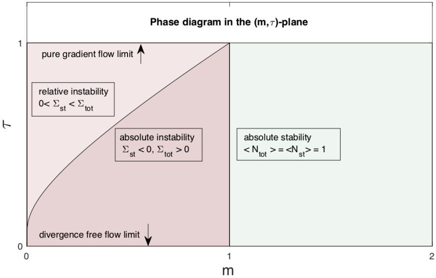
As complexity increases, have a transition from a simple phase portrait to one dominated by unstable equilibria with an admixture of a smaller, but still exp in N, no. of stable equilibria. (Fyodorov & Nadal 2012, Aufinger, Ben Arous & Cerny 2013). Width of the transition region is $1/\sqrt[3]{N}$.

Non-gradient flow: How many equilibria are stable?

Gradient flow calculation doesn't work. New method ? (more on this later)

Claim (Gerárd Ben Arous, Yan Fyodorov, BK) Assume $\tau < 1$ and $m < 1$. Then

$$\langle \mathcal{N}_{st} \rangle \approx e^{N\,\Sigma_{st}(m,\tau)}, \quad \Sigma_{st}(m,\tau) = \Sigma_{tot}(m) - \frac{1+\tau}{2\tau}(1-m)^2.$$



Have $\Sigma_{st} < 0$ under the curve, the prob to find at least one stable EQM is exp small. Put this in the context of May-Wigner instability transition. Also, only a tiny prop of stable EQA above the curve: $p_{st}^{(a)} \approx e^{-N\frac{1+\tau}{2\tau}(1-m)^2}$.

Statistics of unstable directions at equilibrium

Let $\kappa(\mathbf{x}_*)$ be the no. of unstable directions of EQM at $\mathbf{x}_*$, i.e. the no. of EV of $\left(\frac{\partial F_i}{\partial x_j}(\mathbf{x}_*)\right)$ with positive real parts. The instability index $\kappa(\mathbf{x}_*)/N$ is a measure of instability of EQM (think of a walker at a saddle on a random surface)

In the limit $N \gg 1$ the instability index is continuous. Call EQM at $\mathbf{x}_*$ α-stable if $\kappa(\mathbf{x}_*)/N < \alpha$, and denote by $\mathcal{N}_\alpha$ the number of α-stable EQA.

Caution: depending on precision of counting in the limit $N \gg 1$, $\mathcal{N}_0 > \mathcal{N}_{st}$ as $\mathcal{N}_0$ counts EQA with negligible proportion of unstable directions.

Let $m_\alpha \in [-1, 1]$ be the solution of equation $\alpha = \frac{2}{\pi} \int_m^1 \sqrt{1 - t^2}\, dt$ for m.

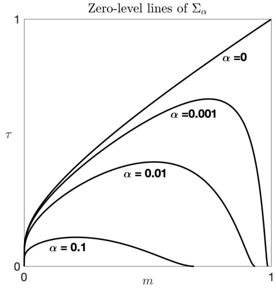
Claim (Gerárd Ben Arous, Yan Fyodorov, BK) Assume $\tau < 1$ and $m < 1$. Then

$$\langle \mathcal{N}_\alpha \rangle \approx e^{N\, \Sigma_\alpha(m,\tau)}, \quad \text{where}$$

$$\Sigma_\alpha(m,\tau) = \Sigma_{tot}(m) - \frac{1+\tau}{2\tau}(m_\alpha - m)^2 \text{ if } 0 < m < m_\alpha \text{ and}$$

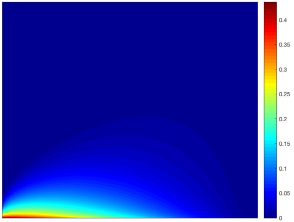
$$\Sigma_\alpha(m,\tau) = \Sigma_{tot}(m) \qquad \text{if } m_\alpha < m < 1.$$

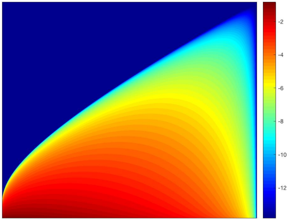
Statistics of unstable directions at equilibrium



On LHS: Zero-level lines of the complexity exponent $\Sigma_\alpha(m,\tau)$.This exponent is negative below the zero-level line and positive above.

Plots below: In the absolute instability regime, $(m,\tau) \mapsto \alpha$ where α is such that that the zero-level line of Σ_α passes through (m,τ). Plots below are heatmaps of $\alpha(m,\tau)$ (left) and $\ln \alpha(m,\tau)$ (right).





Statistics of unstable directions: prob density of EQA with a fixed index $\alpha \in [0,1]$

The prob. for a randomly chosen EQM to have its instability index in the interval (α_1, α_2) is given by $\int_{\alpha_1}^{\alpha_2} \nu(\alpha) d\alpha$, with density

$$\nu(\alpha) = \frac{d}{d\alpha} \left\langle \frac{\mathcal{N}_\alpha}{\mathcal{N}_{tot}} \right\rangle$$

Corollary In the annealed approximation and to leading order in N

$$\nu^{(a)}(\alpha)\left(= \frac{d}{d\alpha}\frac{\langle \mathcal{N}_\alpha \rangle}{\langle \mathcal{N}_{tot} \rangle}\right) = \frac{1}{2}\sqrt{\frac{\pi N(1+\tau)}{2\tau(1-m_\alpha^2)}} e^{-\frac{1+\tau}{2}\left(\frac{\sqrt{N}(m_\alpha - m)}{\sqrt{\tau}}\right)^2} \quad (0 < m < 1)$$

That is, for every $0 < m < 1$, density $\nu^{(a)}(\alpha)$ peaks at $\alpha = \frac{2}{\pi}\int_m^1 \sqrt{1-t^2}\, dt$

No access to the entire transition region between stability and instability, but in the left tail of this region ($m = 1 - \delta/\sqrt{N}$ and $1 \ll \delta \ll \sqrt{N}$):

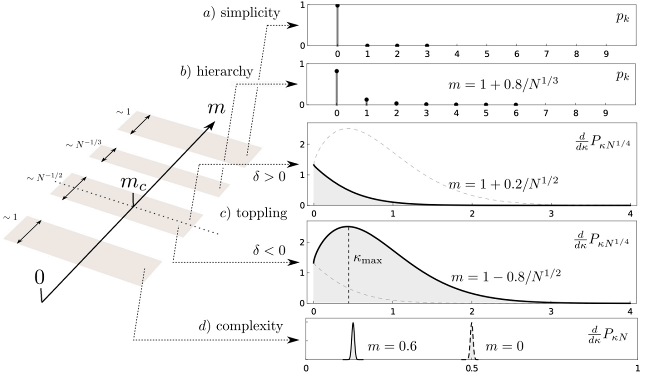
$$\frac{1}{N^{3/4}}\nu^{(a)}\left(\frac{\gamma}{N^{3/4}}\right) \propto e^{-\frac{1+\tau}{2\tau}\left[\delta - \frac{1}{2}\left(\frac{3\pi}{2}\gamma\right)^{2/3}\right]^2}.$$

That is, typical EQA have $N^{1/4}$ unstable directions in the left tail of the TR.

Statistics of unstable directions at equilibrium

Conjecture: no. of unstable directions in the entire TR is prop to $N^{1/4}$

Verified in the annealed approximation for the pure gradient flow (random surface model) in Jacek Grela and BK (in preparation)



Method and assumptions

Our approach uses large deviation theory for RM: estimates on prob of large deviation for x_{max} from its mean value $x_* = 1 + \tau$ and large deviation principle for EV counting measure $\mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{z_j}$ (a lá Sanov) due to Ben Arous and Guionnet 1997 and Ben Arous and Zeitouni 1998.

In the right tail, $\Pr(x > x_*) = e^{-N\Psi(x)+o(N)}$, and consequently

$$\langle \Theta(x - x_{max}) |\det(X - xI)|\rangle_X \approx \langle |\det(X - xI)|\rangle_X$$

and

$$\langle |\det(X - xI)|\rangle_X = \left\langle e^{\sum_j \ln|z_j - x|} \right\rangle \approx e^{N\Phi(x; d\mu_{eq})}$$

where $\Phi(x; d\mu_{eq})$ is the log-potential of the elliptic distribution

$$\Phi(x; d\mu) = \int_{\mathbb{C}} \ln|z - x|\, d\mu(z)$$

Method and assumptions

In the left tail

$$\Pr(x_{max} < x) = e^{-N^2 K_\tau(x) + o(N^2)}, \quad K_\tau(x) = \inf_{\mu \in B_x} \mathcal{J}_\tau[\mu]\,.$$

where $\mathcal{J}_\tau$ is the LDP functional

$$\mathcal{J}_\tau[\mu] = \frac{1}{2}\int_{\mathbb{C}} \left[\frac{(\mathrm{Re}\, z)^2}{1+\tau} + \frac{(\mathrm{Im}\, z)^2}{1-\tau}\right] d\mu(z) - \frac{1}{2}\int_{\mathbb{C}^2} \log|z-w|\, d\mu(z)d\mu(w) - \frac{3}{8}$$

and B_x is the set of prob meas in $\mathbb{C}$ with supp to the left of the line $\Re z = x$.

From this one obtains factorisation

$$\langle \Theta(x - x_{max}) |\det(X - xI)|\rangle_X = e^{N\Phi(x; d\mu_x) + o(N)} \Pr(x_{max} < x),$$

where μ_x is the minimiser of J_τ on B_x, and we are getting into a delicate situation. Can complete our analysis under two assumptions

(i) $\Phi(x; d\nu_x)$ is continuous in x at $x = x_*$; and (ii) the sub-leading term in the LD prob in the right tail is of order N.

Open questions

- Statistics of unstable directions in the transition region?
- Is the average value representative? Magnitude of deviations from the av. value?
- Is the annealed approximation accurate? Can one work out true probabilities for an EQM to have fixed number of unstable directions
- Cycles?
- Signature of the May-Wigner instability transition in the system dynamics?
- Challenging random matrix problem, analogue of Tracy-Widom for x_{max} in the real Ginibre/elliptic ensemble ?

THANK YOU