

# A Direct Quadrature Based Nonlinear Filtering with Extended Kalman Filter Update for Orbit Determination

Jangho Yoon, Yunjun Xu and Prakash Vedula

**Abstract**—An optimal estimation of the states of a nonlinear continuous system with discrete measurements can be achieved through the solution of the Fokker-Planck equation, along with the Bayes' formula. However, solving the Fokker-Planck equation is restrictive in most cases. Recently a nonlinear filtering algorithm using a direct quadrature method of moments and the extended Kalman filter update mechanism was proposed, in which the associated Fokker-Planck equation was solved efficiently and accurately via discrete quadrature and the measurement update was done through the extended Kalman filter update mechanism. In this paper this hybrid filter based on the DQMOM and the EKF update is applied to the orbit determination problem with appropriate modification to mitigate the filter smugness. Unlike the extended Kalman filter, the hybrid filter based on the DQMOM and the EKF update does not require the burdensome evaluation of the Jacobian matrix and Gaussian assumption for system noise, and can still provide more accurate estimation of the state than those of the extended Kalman filter especially when measurements are sparse. Simulation results indicate that the advantages of the hybrid filter based on the DQMOM and the EKF update make it a promising alternative to the extended Kalman filter for orbit estimation problems.

**Index Terms**—Nonlinear filtering, Fokker-Planck equation, orbit determination, DQMOM.

## I. INTRODUCTION

Estimating the state of a dynamic system from noisy observations is very important in engineering. This problem has been the subject of considerable research interest ever since the time Gauss formulated the deterministic least-square technique for simple orbit determination. Up to date, many different techniques have been developed and used in a wide variety of applications, such as navigation and guidance systems, radar tracking, sonar ranging, satellite and airplane states determination, and the volatility of financial system estimation using stock market data, etc. [1][2][3][4][5].

The Bayesian framework is the most commonly used optimal nonlinear filtering methodology [6][7][8], in which the principle is to find the probability density function (PDF) of the state conditioned on the history of the measurements. The extended Kalman filter (EKF) is most commonly used nonlinear Bayesian estimator based on the assumptions that are 1) perturbations from the mean trajectory are small, and

2) the conditional density function of the state is Gaussian. When these assumptions are violated (especially the first one), the EKF performs poorly or becomes unstable. These problems can be overcome by solving the filtering problem in the more general setting of nonlinear systems that the previously mentioned assumptions are not needed.

Estimation theory for a more general setting of nonlinear systems has been established since the 1960s [1][9][10]. The exact nonlinear filter for systems with continuous nonlinear dynamics and discrete nonlinear observations consists of two equations (Fig. 1) [1]. A partial differential equation called the Fokker-Planck equation (FPE) [11][12] describes how the conditional density evolves between measurements, and the Bayes' formula describes how the conditional density is modified by information coming from measurements.

Normally, it is not an easy task to achieve a closed form solution of the FPE with only a few exceptions. As a result, it usually has to be evaluated numerically. In recent years thanks to the advance in computer technology and numerical methods this estimation technique has been used in target tracking problems[2][3][13] and the relative position estimation of a satellite[14][15]. These researchers employed efficient numerical schemes such as the alternating direction implicit method with adaptive moving grids to alleviate the high computational cost [13][14][15]. However, as shown in these papers, the computational cost is still high.

A nonlinear filtering method using the direct quadrature method of moments, along with Bayesian update of the conditional state PDF was first proposed by Xu and Vedula [16]. This approach involves a representation of the state conditional PDF in terms of a finite summation of the Dirac delta functions. Using a small number of scalars (in the Dirac delta function), the method is able to efficiently and accurately model stochastic processes through a set of ordi-

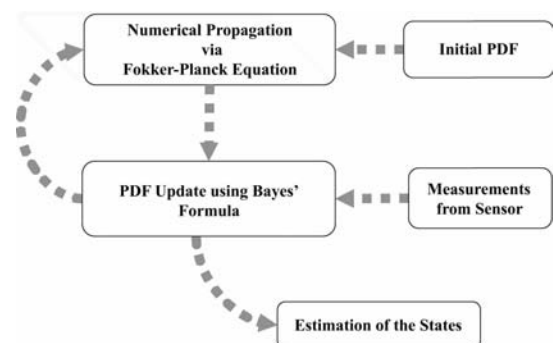


Fig. 1: FPE based nonlinear estimation

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nary differential equations (ODEs). The DQMOM approach could lead to a significant reduction in computational cost, compared to finite difference (and other equivalent) methods, especially for high dimensional problems. Further studies on the DQMOM base nonlinear filter reveal that the ‘‘degeneracy’’ phenomenon, similar to the one exists in a typical particle filter, occasionally appears because in this algorithm only the weight is updated and the abscissas remain the same [17]. The un-updated abscissas might be propagated into the tail locations of the PDF where no significant statistical meaning is carried. This problem is solved by employing the update step from the EKF or Unscented Kalman Filter (UKF) (so that the linearized model is not required in the update stage of the filter)[18]. Note that similar approaches of combing different filtering strategies have been proposed, such as KF and UKF by Sadhua [19], and FPE and EKF/UKF by Daum [20]. The purpose of this paper is to investigate and use the hybrid filter based on the DQMOM and the EKF update for accurate sequential orbit determination of an earth-orbiting satellite or space object.

When the noise inputs to the system and measurement noise are small such as orbit determination, EKF can become ‘‘smug.’’ Smug means the covariance matrix  $\mathbf{P}_k$  becomes so small and as a result the gain becomes very small. Due to the small gain the filter becomes over confident in its estimation and refuse to accept new information from the measurements[21]. The same problem was observed with The hybrid filter based on the DQMOM and the EKF update. Several solutions have been devised to deal with this problem [1][21][22]. There are two different approaches, adaptive and non-adaptive [21]. In this work a non-adaptive approach of using fixed error covariance matrix is employed to keep the gain becoming too small.

This paper is divided as follows. In Section II, we briefly describe the framework of the nonlinear filtering using the FPE approach. In Section III, the DQMOM approach is summarized. Section IV includes the measurement update using the EKF update formula and method to handle the filter smug. Finally a numerical simulation of orbit determination problem is presented followed by the discussion of the results and the conclusion.

## II. FOKKER-PLANCK EQUATION AND NONLINEAR ESTIMATION

In this section we state the nonlinear filtering problem to be addressed in the work. A nonlinear dynamic systems can be modeled as an n-dimensional continuous Itô stochastic differential equation (SDE):

$$d\mathbf{x}_t = \mathbf{F}(\mathbf{x}_t, t)dt + \mathbf{G}(\mathbf{x}_t, t)d\beta_t \quad (1)$$

where  $\mathbf{x}_t \in \mathbb{R}^{n \times 1}$ ,  $\mathbf{F}(\mathbf{x}_t, t) \in \mathbb{R}^{n \times 1}$  and  $\mathbf{G}(\mathbf{x}_t, t) \in \mathbb{R}^{n \times m}$  are the state vector, state function, and diffusion matrix, respectively.  $\{d\beta_t\} \in \mathbb{R}^{n \times 1}$  is a Brownian motion process with  $E\{d\beta_t d\beta_t^T\} = \mathbf{Q}(t)dt$ . The measurement  $\mathbf{y}_{t_k}$  taken at the discrete time  $t_k$  is defined as

$$\mathbf{y}_{t_k} = \mathbf{h}(\mathbf{x}_{t_k}, t_k) + \mathbf{v}_{t_k} \quad (2)$$

where  $\mathbf{h}(\mathbf{x}_{t_k}, t_k) \in \mathbb{R}^{q \times 1}$  is the measurement function and  $\mathbf{v}(t_k)$  is the measurement noise, which is assumed to be a Gaussian white noise with a covariance matrix of  $\mathbf{R}(t)$  [1].

If the process described by the SDE is a Markovian diffusion process, the PDF that characterizes this process is governed by the Fokker-Planck equation [1].

$$\frac{\partial p}{\partial t} = - \sum_{i=1}^n \frac{\partial [p\mathbf{F}_i]}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 [p(\mathbf{G}\mathbf{Q}\mathbf{G}^T)_{ij}]}{\partial x_i \partial x_j} \quad (3)$$

where  $p$  is the conditional state PDF,  $p(\mathbf{x}_t | \mathbf{Y}_{t_{k-1}})$  and  $\mathbf{Y}_{t_{k-1}}$  is the measurement history vector defined as  $\mathbf{Y}_{t_{k-1}} \triangleq [\mathbf{y}(0), \mathbf{y}(2), \dots, \mathbf{y}(k-1)]^T$ . Also  $p(\mathbf{x}_t | \mathbf{Y}_{t_{k-1}})$  can be considered as the probability density function of a process governed by the Itô SDE between measurements ( $t_{k-1} < t < t_k$ ), and found by solving (3) between  $t_{k-1}$  and  $t_k$ .

Once the PDF function,  $p(\mathbf{x}_{t_k}, t | \mathbf{Y}_{t_{k-1}})$  is found from (3) and the measurement  $\mathbf{y}_{t_k}$  made at the time instant  $t_k$ , the Bayes’ formula is used to update the conditional PDF to  $p(\mathbf{x}_{t_k} | \mathbf{Y}_{t_k})$ , where  $\mathbf{Y}_{t_k} = [\mathbf{y}(0), \dots, \mathbf{y}(t_k)]^T$ .

$$p(\mathbf{x}_{t_k} | \mathbf{Y}_{t_k}) = \frac{p(\mathbf{x}_{t_k} | \mathbf{Y}_{t_{k-1}}) p(\mathbf{y}_{t_k} | \mathbf{x}_{t_k})}{\int p(\xi_{t_k} | \mathbf{Y}_{t_k}) p(\mathbf{y}_{t_k} | \xi_{t_k}) d\xi} \quad (4)$$

The PDF  $p(\mathbf{y}_{t_k} | \mathbf{x}_{t_k})$  is defined by the characteristics of the sensor and is usually assumed to have a Gaussian distribution

$$p(\mathbf{y}_{t_k} | \mathbf{x}_{t_k}) = \frac{1}{(2\pi)^{q/2} |\mathbf{R}|^{1/2}} e^{-\frac{1}{2} [\mathbf{y}_{t_k} - \mathbf{h}(\mathbf{x}_{t_k})]^T \mathbf{R}^{-1} [\mathbf{y}_{t_k} - \mathbf{h}(\mathbf{x}_{t_k})]} \quad (5)$$

The combination of FPE and Bayes’ formula mentioned above builds a two-step process to obtain the desired conditional PDF  $p(\mathbf{x}_{t_k} | \mathbf{Y}_{t_k})$  [1]. After  $p(\mathbf{x}_{t_k} | \mathbf{Y}_{t_k})$  is obtained from (4) the state estimation can be made by calculating the following integral

$$\hat{x}_i(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} x_i p(\mathbf{x}, t) \prod_{j=1}^n dx_j \quad (6)$$

Hence, using the updated conditional PDF, the minimum mean-square estimate (MMSE) estimates of any state variables or functions of state variables can be obtained. The central issue associated with the FPE based nonlinear filtering technique is the high computational cost.

## III. DIRECT QUADRATURE METHOD OF MOMENTS

The detailed derivation of the direct quadrature method of moments (DQMOM) with a systematic initialization can found in [16][18] and here only a brief outline of the algorithm is presented.

### A. Summary of DQMOM

DQMOM is an approximation method that can solve FPE efficiently and accurately. In the DQMOM approach, the state or state conditional PDF  $p = p(\mathbf{x}_t | \mathbf{Y}_{t_k})$  is approximated by the summation of a weighted multi-dimensional Dirac delta function as

$$p = \sum_{\alpha=1}^N w_{\alpha} \prod_{j=1}^{N_s} \delta[x_j - \langle x_j \rangle_{\alpha}] \quad (7)$$

where  $w_\alpha$ ,  $\alpha = 1, \dots, N$  is the corresponding weight for node  $\alpha$ , and  $\langle x_j \rangle_\alpha$ ,  $\alpha = 1, \dots, N$ ;  $j = 1, \dots, N_s$  is the property vector of node  $\alpha$  called "abscissas."  $N$  is the number of nodes used in the PDF representation. In this representation, there are total  $N(N_s + 1)$  unknown variables which will be solved through the moment constraints. The weighted abscissas  $\zeta_{j\alpha} \equiv W_\alpha \langle x_j \rangle_\alpha$  is introduced so that the moments of (3) can be derived as

$$\begin{aligned} \bar{S}_{k_1, \dots, k_{N_s}} &= \sum_{\alpha=1}^N \left[ \left( 1 - \sum_{j=1}^{N_s} k_j \right) \prod_{k=1}^{N_s} \langle x_k \rangle_\alpha^{k_k} \right] a_\alpha \\ &+ \sum_{\alpha=1}^N \sum_{j=1}^{N_s} k_j \langle x_j \rangle_\alpha^{k_j-1} \prod_{k=1, k \neq j}^{N_s} \langle x_k \rangle_\alpha^{k_k} b_{j\alpha} \end{aligned} \quad (8)$$

where  $w_\alpha$ ,  $\zeta_{j\alpha}$ , and  $\delta_{j\alpha}$  are functions of time and they can be propagated in a fairly quick fashion based upon the set of algebraic ODEs (8) as compared to its original PDE version (3). In these ODEs  $dw_\alpha/dt \equiv a_\alpha$  and  $\zeta_{j\alpha}/dt \equiv b_{j\alpha}$ .  $\bar{S}_{k_1, \dots, k_{N_s}} \equiv \bar{S}_{k_1, \dots, k_{N_s}}^1 + \bar{S}_{k_1, \dots, k_{N_s}}^2$ . The derivation of  $\bar{S}_{k_1, \dots, k_{N_s}}^1$  and  $\bar{S}_{k_1, \dots, k_{N_s}}^2$  can be found in [16] as

$$\begin{aligned} \bar{S}_{k_1, \dots, k_{N_s}}^1 &= \sum_{i=1}^{N_s} \sum_{\alpha=1}^N [k_i w_\alpha(t) \langle x_1 \rangle_\alpha^{k_1} \dots \langle x_{i-1} \rangle_\alpha^{k_{i-1}} \langle x_i \rangle_\alpha^{k_i} \\ &\langle x_{i+1} \rangle_\alpha^{k_{i+1}} \dots \langle x_{N_s} \rangle_\alpha^{k_{N_s}} \mathbf{F}_i(\langle x_1 \rangle_\alpha \dots \langle x_{N_s} \rangle_\alpha)] \end{aligned} \quad (9)$$

and

$$\begin{aligned} \bar{S}_{k_1, \dots, k_{N_s}}^2 &= \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \sum_{\alpha=1}^N \left[ \frac{w_\alpha k_i k_j \prod_{k=1}^{N_s} \langle x_k \rangle_\alpha^{k_k}}{\langle x_i \rangle_\alpha \langle x_j \rangle_\alpha [D(\mathbf{x})]_{ij} \langle x_1 \rangle_\alpha \dots \langle x_{N_s} \rangle_\alpha} \right] \text{if } i \neq j \\ \bar{S}_{k_1, \dots, k_{N_s}}^2 &= \sum_{\alpha=1}^N \left[ \frac{w_\alpha k_i (k_i - 1) \prod_{k=1}^{N_s} \langle x_k \rangle_\alpha^{k_k}}{\langle x_i \rangle_\alpha^2 [D(\mathbf{x})]_{ij} \langle x_1 \rangle_\alpha \dots \langle x_{N_s} \rangle_\alpha} \right] \text{if } i = j \end{aligned}$$

Once the abscissas and weights are propagated from (8), any selected statistical moment of the state PDF, such as mean, variance, and covariance, etc., can be calculated from

$$M^{k_1, k_2, \dots, k_{N_s}} = \sum_{\alpha=1}^N w_\alpha \prod_{j=1}^{N_s} \langle x_j \rangle_\alpha^{k_j} \quad (11)$$

where  $k_1, k_2, \dots, k_{N_s}$  are nonnegative integers and used to denote the  $k_1, k_2, \dots, k_{N_s}^{\text{th}}$  moments of the state statistics. The steps to propagating the FPE are tabulated in Table I.

TABLE I: Propagation of the FPE through DQMOM

Step 1: Initialization of $w_\alpha$ and $\langle x_j \rangle_\alpha$ , $\alpha = 1, \dots, N$ , $j = 1, \dots, N_s$ at time $t_0$
Step 2: Between the measurements $t_i$ and $t_{i+1}$ , $i = 0, 1, \dots$
Step 3: Calculate $\bar{S}_{k_1, \dots, k_{N_s}}$ using (9) - (10).
Step 4: Propagate (8) from $t_i$ to $t_{i+1}$
Step 5: Calculate the user specified characteristic moments according to (11).
Step 6: If $t_{i+1} < t_f$ , go to Step 2.

## B. Initialization

The initially selected abscissas and weight have to represent the desired initial statistical information such as the mean and the variance correctly. Hence, proper modifications of the weights and abscissas are usually necessary. The desired mean and variance values of the state  $j$  can constructed from a set of chosen abscissas and weights as

$$\mu_j = \sum_{\alpha=1}^N w_\alpha \langle x_j \rangle_\alpha, \quad j = 1, \dots, N_s \quad (12)$$

$$\sigma_j^2 = E[\langle x_j \rangle_\alpha^2] - \mu_j^2 = \sum_{\alpha=1}^N w_\alpha \langle x_j \rangle_\alpha^2 - \mu_j^2, \quad j = 1, \dots, N_s \quad (13)$$

The mean can be modified by adding a constant to each abscissa and the variance can be adjusted by multiplying a constant to each abscissas while the weights are kept same. Let us assume that the modified abscissas are

$$\langle x_j \rangle_{\alpha, \text{mod}} = C_{1j} \langle x_j \rangle_\alpha + C_{2j} \quad j = 1, \dots, N_s \quad (14)$$

and the desired mean and variance values are  $\mu_{j,d}$  and  $\sigma_{j,d}^2$ . In order to maintain the desired mean and variance values, the coefficients in (14) are

$$C_{1j} = \frac{\sigma_{i,d}}{\sigma_j} \text{ and } C_{2j} = \mu_{j,d} - \mu_j \frac{\sigma_{i,d}}{\sigma_j} \quad j = 1, \dots, N_s \quad (15)$$

By substituting  $C_{1j}$  and  $C_{2j}$ , the adjusted abscissas with desired mean and variance is

$$\langle x_j \rangle_{\alpha, \text{mod}} = \frac{\sigma_{i,d}}{\sigma_j} \langle x_j \rangle_\alpha + \mu_{j,d} - \mu_j \frac{\sigma_{i,d}}{\sigma_j} \quad j = 1, \dots, N_s \quad (16)$$

## IV. UPDATE MECHANISM

In this section, the update mechanisms employed to mitigate the "degeneracy" phenomenon and prevent the filter from becoming smug will be described.

### A. Update through the Extended Kalman Filter

The prediction of the states  $\hat{\mathbf{x}}^- = [\hat{x}_i^-]_{i=1, \dots, N_s}$  and the error covariance matrix  $\mathbf{P}^- = [P_{ij}^-]_{i=1, \dots, N_s, j=1, \dots, N_s}$  at time step  $t_k$  are calculated from the abscissas and weights propagated through the DQMOM(8) as

$$\hat{x}_i^- = \sum_{\alpha=1}^N w_\alpha \langle x_i \rangle_\alpha \quad i = 1, \dots, N_s \quad (17)$$

and

$$P_{ij}^- = \left[ \sum_{\alpha=1}^N w_\alpha \langle x_i \rangle_\alpha \langle x_j \rangle_\alpha \right] - \hat{x}_i^- \hat{x}_j^- \quad (18)$$

The updated Kalman gain  $\mathbf{K}_k$  at the time step  $t_k$  is given by

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T [\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}]^{-1} \quad (19)$$

where the linearized measurement model is given by  $\mathbf{H}_k = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}|_{\mathbf{x}=\mathbf{x}^-}$ . The estimation of the state and the estimated error covariance are given as

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k (\mathbf{y}_k - \hat{\mathbf{y}}_k) \quad (20)$$

and

$$\mathbf{P}_k^+ = [\mathbf{I} - \mathbf{K}_k \mathbf{H}_k] \mathbf{P}_k^- \quad (21)$$

After the measurement update, the abscissas are re-sampled to match the updated mean and covariance. The procedure is the same as the initialization strategy described in Section III.B.

### B. Dealing with the filter smugness

“Smug” means the error covariance matrix  $\mathbf{P}_k$  becomes so small and as a result the gain becomes very small. Under this situation the filter believes that the states are well known and ignores the measurements from the sensor [21].

Several solutions have been devised to deal with this problem [1][21][22], and they can be categorized into two different approaches, adaptive and non-adaptive [21]. In this work a non-adaptive approach is employed to keeping the gain from becoming too small. Instead of using the covariance  $\mathbf{P}_k^-$  from DQMOM, a fixed  $\bar{\mathbf{P}}$  is used. Thus, (19) becomes

$$\mathbf{K}_k = \bar{\mathbf{P}}\mathbf{H}_k^T[\mathbf{H}_k\bar{\mathbf{P}}\mathbf{H}_k^T + \mathbf{R}]^{-1} \quad (22)$$

Appropriate fixed covariance  $\bar{\mathbf{P}}$  can be found/designed through extensive simulation [1].

## V. PROBLEM DESCRIPTION

### A. Process Model

The equations of motion that governs the motion of a satellite in a low earth orbit is [21]:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{a}_G + \mathbf{a}_D \quad (23)$$

where  $\mu$  and  $\mathbf{r} = [x, y, z]^T$  are the gravitational parameter and the position vector, respectively. The scalar  $r$  is the magnitude of  $\mathbf{r}$ , i.e.,  $r = \sqrt{x^2 + y^2 + z^2}$ .  $\mathbf{a}_D$  represent the drag due to the earth atmosphere and is proportional to the atmospheric density  $\rho$  and the square of the velocity relative to the atmosphere. The Earth is not a spherically symmetric body but bulged at the equator, and is also generally asymmetric. As a result, the gravitational field around the earth is not isometric. The  $\mathbf{a}_G$  is the perturbation due to this uneven gravitational field. In this work,  $\mathbf{a}_D$  and  $\mathbf{a}_G$  are considered as a part of process noise to the system. So, the nominal process equation used in this work is

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \quad (24)$$

### B. Fokker-Planck equation of the Keplerian equation

The state-space form of the Keplerian equation of motion with the process noise is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{\mu}{r^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu}{r^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu}{r^3} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \mathbf{w}_t \triangleq \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{w}_t \quad (25)$$

where  $\mathbf{w}_t$  is a white Gaussian noise process with  $E[\mathbf{w}_t\mathbf{w}_t^T] = \mathbf{Q}_t\delta(t-\tau)$

The corresponding Fokker-Planck equation can be found by substituting (25) into (3) as

$$\begin{aligned} \frac{\partial p}{\partial t} = & - \left( \frac{\partial p}{\partial x}\dot{x} + \frac{\partial p}{\partial y}\dot{y} + \frac{\partial p}{\partial z}\dot{z} \right) + \frac{\mu}{r^3} \left( \frac{\partial p}{\partial \dot{x}}x + \frac{\partial p}{\partial \dot{y}}y + \frac{\partial p}{\partial \dot{z}}z \right) \\ & + \frac{Q_4}{2} \frac{\partial^2 p}{\partial \dot{x}^2} + \frac{Q_5}{2} \frac{\partial^2 p}{\partial \dot{y}^2} + \frac{Q_6}{2} \frac{\partial^2 p}{\partial \dot{z}^2} \end{aligned} \quad (26)$$

where  $Q_i$ ,  $i = 4, 5, 6$  are the last three diagonal members of  $[\mathbf{G}\mathbf{Q}\mathbf{G}^T]$  from (3)

### C. Measurement Model

The inertial position vector of a satellite can be written as the sum of the range vector and the radar site position vector [21] as

$$\mathbf{r} = \mathbf{R}_s + \boldsymbol{\rho} \quad (27)$$

where  $\mathbf{R}_s$  is the position of the sensor, and  $\boldsymbol{\rho} = [\rho_e, \rho_e, \rho_n]^T$  is the position vector of a satellite in the local/sensor coordinate as

$$\boldsymbol{\rho} = \rho_u\hat{\mathbf{u}} + \rho_e\hat{\mathbf{e}} + \rho_n\hat{\mathbf{n}} \quad (28)$$

wherein the subscriptions,  $u$ ,  $e$ , and  $n$  stand for “zenith”, “east”, and “north”, respectively.

The measurements are range, azimuth and elevation. The range  $\rho$  can be found from

$$\rho = \sqrt{\rho_u^2 + \rho_e^2 + \rho_n^2} \quad (29)$$

The azimuth and elevation angles are

$$az = \tan^{-1} \left( \frac{\rho_e}{\rho_n} \right) \quad \text{and} \quad el = \tan^{-1} \left( \frac{\rho_u}{\sqrt{\rho_e^2 + \rho_n^2}} \right) \quad (30)$$

For the sensor position vector  $\mathbf{R}_s$ , it is advisable to account for the precise shape of the Earth to avoid large errors [21].  $\mathbf{R}_s$  in the geocentric inertia coordinate accounted for the Earth’s equatorial bulge and its magnitude can be found by [23]

$$\begin{aligned} \mathbf{R}_s &= r_\delta \cos \theta \mathbf{I} + r_\delta \sin \theta \mathbf{J} + r_k \mathbf{K} \\ \|\mathbf{R}_s\| &= \sqrt{r_\delta^2 + r_k^2} \end{aligned} \quad (31)$$

where  $\theta$  is the sidereal time of the sensor (local sidereal time). Calculation of  $\theta$  with the Greenwich sidereal time (GST) at the beginning of the particular year can be found from Astronomical Almanac.[21][23], and  $r_\delta$  and  $r_k$  can be calculated using

$$\begin{aligned} r_\delta &= \left[ \frac{R_\oplus}{\sqrt{1 - e_\oplus^2 \sin^2 \lambda}} + H \right] \cos \lambda \\ r_k &= \left[ \frac{R_\oplus(1 - e_\oplus^2)}{\sqrt{1 - e_\oplus^2 \sin^2 \lambda}} + H \right] \sin \lambda \end{aligned} \quad (32)$$

where  $\lambda$  is the geodesic latitude of the sensor location, and  $R_\oplus = 6378.1363\text{km}$  and  $e_\oplus = 0.081819221456$  are the mean equatorial radius of the Earth and the eccentricity of the Earth, respectively.  $H$  is the elevation above the sea level.

The position vector  $\boldsymbol{\rho}$  in the inertial coordinate frame is

$$\boldsymbol{\rho} = \begin{bmatrix} x - \|\mathbf{R}_s\| \cos \lambda \cos \theta \\ y - \|\mathbf{R}_s\| \cos \lambda \sin \theta \\ z - \|\mathbf{R}_s\| \sin \lambda \end{bmatrix} \quad (33)$$

The conversion from the inertial to the sensor coordinate system is given by the following rotation matrix

$$\mathbf{C} = \begin{bmatrix} \cos \lambda & 0 & \sin \lambda \\ 0 & 1 & 0 \\ -\sin \lambda & 0 & \cos \lambda \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (34)$$

So the vector  $\boldsymbol{\rho}$  is

$$\begin{bmatrix} \rho_u \\ \rho_e \\ \rho_n \end{bmatrix} = \begin{bmatrix} \cos \lambda \cos \theta & \cos \lambda \sin \theta & \sin \lambda \\ -\sin \theta & \cos \theta & 0 \\ -\sin \lambda \cos \theta & -\sin \lambda \sin \theta & \cos \lambda \end{bmatrix} \boldsymbol{\rho} \quad (35)$$

## VI. NUMERICAL SIMULATION

In this section the performance of the proposed nonlinear filter, The hybrid filter based on the DQMOM and the EKF update is demonstrated through simulation and compared with the result from EKF. This simulation scenario is based on the one used by Lee and Alfriend [24].

### A. Simulation Scenario

The satellite under consideration has the following orbit parameters:  $a = 6778.136 \text{ km}$ ,  $e = 1.0 \times 10^{-5}$ ,  $i = 51.6^\circ$ ,  $\omega = 30^\circ$ , and  $\Omega = 25^\circ$ . The  $J_2$  and drag perturbation ( $\mathbf{a}_G$  and  $\mathbf{a}_D$  from (23)) are considered as the noise to the system. The location of the sensor is chosen to be the Eglin Air Force Base with  $30.2316^\circ$  latitude and  $86.2147^\circ$  west longitude. The measurement errors are assumed to be Gaussian random processes with zero means and variances of  $\sigma_{range} = 25.0 \text{ m}$ ,  $\sigma_{azimuth} = 0.015^\circ$ , and  $\sigma_{elevation} = 0.015^\circ$ , respectively.

The true initial values of the state vector are set to be  $x_0 = 4011.5713 \text{ km}$ ,  $y_0 = 4702.6493 \text{ km}$ ,  $z_0 = 3238.3582 \text{ km}$ ,  $\dot{x} = -5.653084 \text{ km/s}$ ,  $\dot{y}_0 = 1.5401902 \text{ km/s}$ , and  $\dot{z} = 4.7765408 \text{ km/s}$ . For the filter, the initial values are obtained using the Herrick-Gibbs method [24], and they are  $x_0 = 3931.3399 \text{ km}$ ,  $y_0 = 4608.5963 \text{ km}$ ,  $z_0 = 3173.5911 \text{ km}$ ,

$\dot{x} = -5.540022 \text{ km/s}$ ,  $\dot{y}_0 = 1.5093864 \text{ km/s}$ , and  $\dot{z} = 4.6810100 \text{ km/s}$ .

As stated above, the acceleration due to  $J_2$ , which is approximately  $10^{-5} \text{ km/s}^2$  at low earth orbits is considered as the noise to the system. So the process noise covariance matrix  $\mathbf{Q}(t)$  is set to  $\text{diag}([0 \ 0 \ 0 \ 10^{-10} \ 10^{-10} \ 10^{-10}])$

The simulation of the Keplerian dynamic through the DQMOM is done using canonical unit instead of the standard SI unit [23], which means (24) is nondimensionalized for a better numerical stability. The initial position is used as the distance unit (DU). The velocity unit (VU) is found by  $\sqrt{\mu/\text{DU}}$ , so the time unit (TU) is naturally equal to DU/VU. The measurement update was done in SI unit.

The number of the nodes used for this study was two, and the moment constraints are chosen so that the the first three moments of the PDF were preserved. The details regarding how to set the moment constraints can be found from [16].

### B. Simulation Results

Simulations are executed with two different measurement update frequencies, 1Hz and 0.05Hz, and the result from both the hybrid filter based on the DQMOM and the EKF update and EKF are presented. The figures are the root mean square errors (RMSE) of the position and the velocity estimation produced by Monte Carlo simulation of 30 runs. When the measurement update is frequent (1Hz), as shown in both Fig. 2a and Fig. 3a, The hybrid filter based on the DQMOM and the EKF update performs better in terms of quicker convergence in velocity estimation and better estimation accuracy in both position and velocity estimation. When the time between measurement update is increased to twenty seconds (0.05Hz update frequency), the hybrid filter based on the DQMOM and the EKF update shows quicker convergence and better estimation accuracy in both position and velocity estimation than the EKF as shown in both Fig. 2b and Fig. 3b. Unlike the position RMSE curve shown in Fig.2b, the velocity RMSE curve is smooth and not zigzagged. This is the result of using a fixed error covariance matrix.

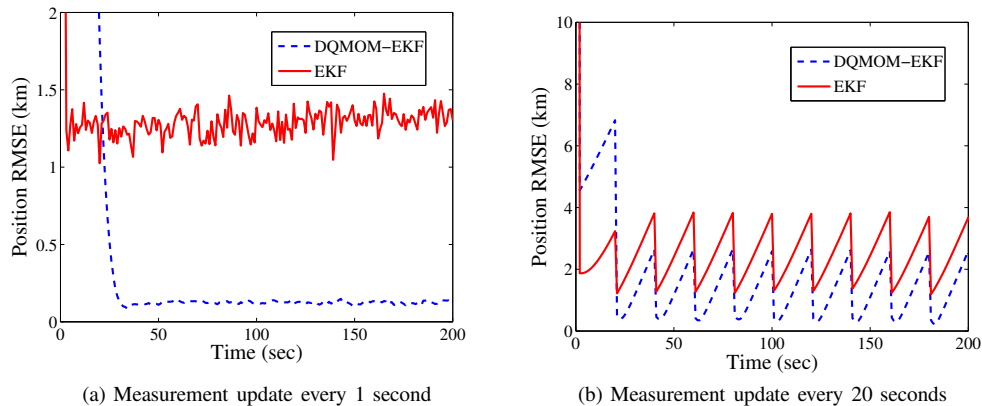


Fig. 2: The RMSE of the absolute magnitude of the position with different measurement update delay

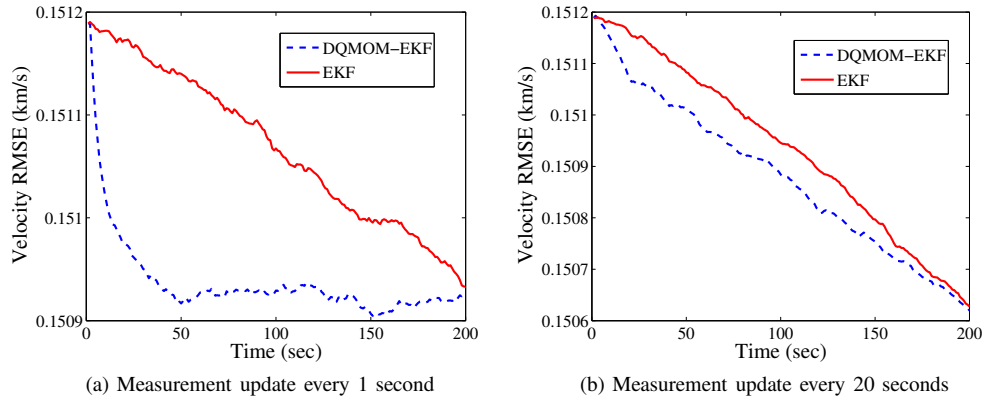


Fig. 3: The RMSE of the Absolute magnitude of the velocity with different measurement update delay

Using a workstation with the Intel<sup>®</sup> 2.33 GHz Xeon processor and MatLab<sup>®</sup>, it takes roughly 70 seconds in CPU time to finish the 200-second simulation for the hybrid filter based on the DQMOM and the EKF update, while the EKF did it in about 0.65 seconds. However, as comparing with other numerical approaches used in nonlinear filtering design, such as in [13][15][18], the computational cost is dramatically reduced and very close to the real-time needs.

## VII. CONCLUSIONS

In this paper a nonlinear filtering algorithm utilizing the DQMOM and the EKF measurement update is used to obtain accurate and efficient orbit estimation. In addition, a fixed error covariance matrix is used for the gain calculation to prevent the filter becoming smug. Nondimensionalized system equation is used for DQMOM to achieve a more stable propagation of the conditional PDF. Simulation results indicate that the performances of the hybrid filter based on the DQMOM and the EKF update is superior to the standard extended Kalman filter for both frequent and sparse measurement update in terms of the estimate accuracy and convergence rate. Also the filter provides the flexibility of implementation without Jacobian matrix of the dynamic equation. The advantages of the proposed nonlinear filtering algorithm show its potential to be suitable for efficient real-time satellite orbit determination.

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