Numerical computation of cell motion in a branching flow

M. G. Blyth

University of East Anglia, Norwich

Capillary Flow Meeting, University of Warwick
1st June 2010

Joint work with:

Hugh Woolfenden (University of East Anglia)

Outline

Outline

Red blood cells and capillaries

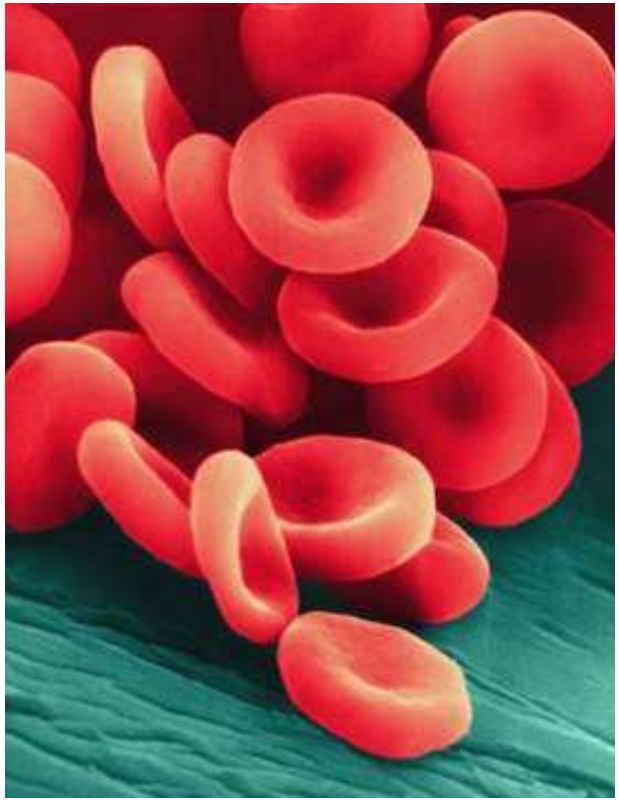


Figure: Healthy RBCs

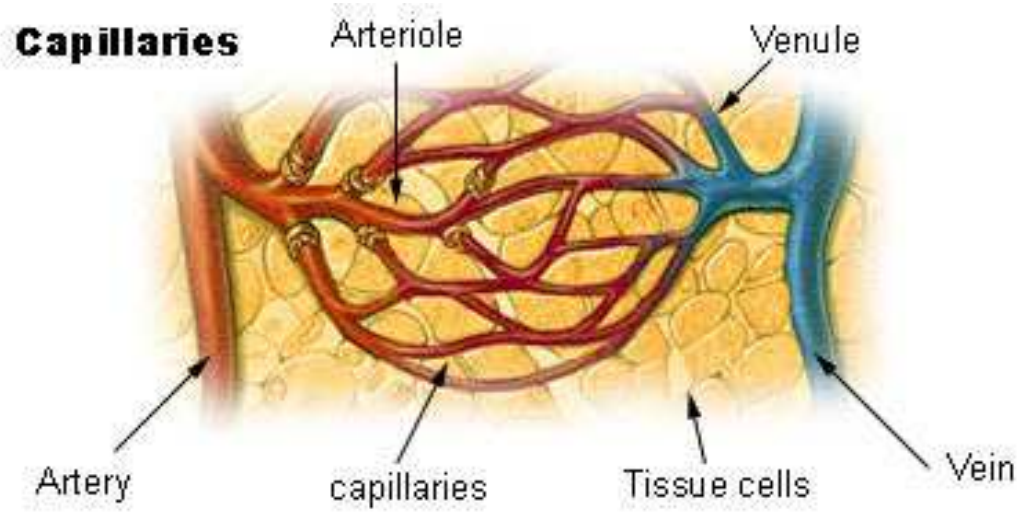


Figure: Capillary bed

Capillary flow

Navier-Stokes equations

$$\rho\Big(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\Big) = -\nabla p + \mu\nabla^2\mathbf{u}$$

$$\nabla\cdot\mathbf{u} = 0$$

Typical Reynolds number:

$$R = \frac{\rho U a}{\mu} \approx 0.001$$

Capillary flow

Stokes equations

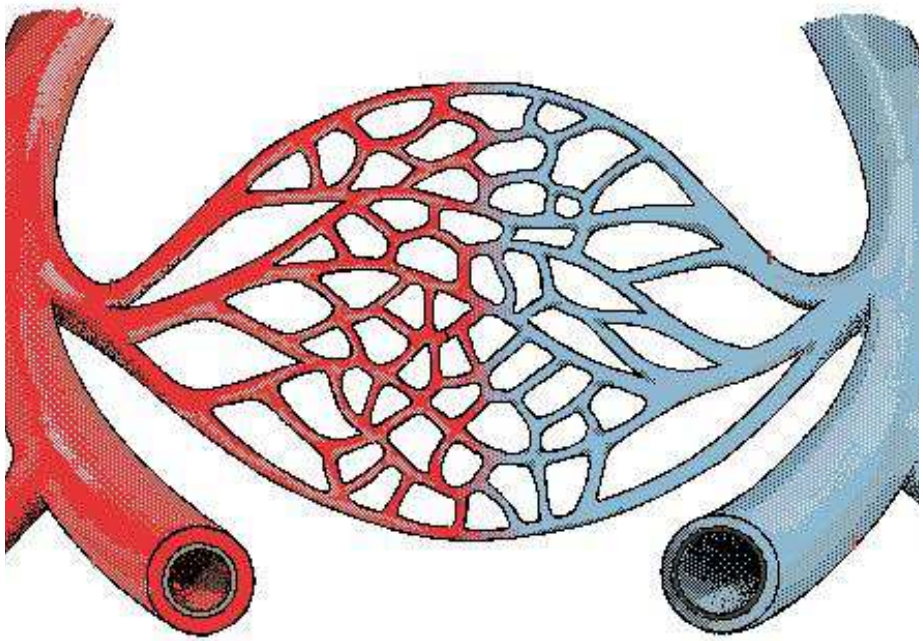
$$0 = -\nabla p + \mu \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

Typical Reynolds number:

$$R = \frac{\rho U a}{\nu} \approx 0.001$$

Capillary Network



Network models

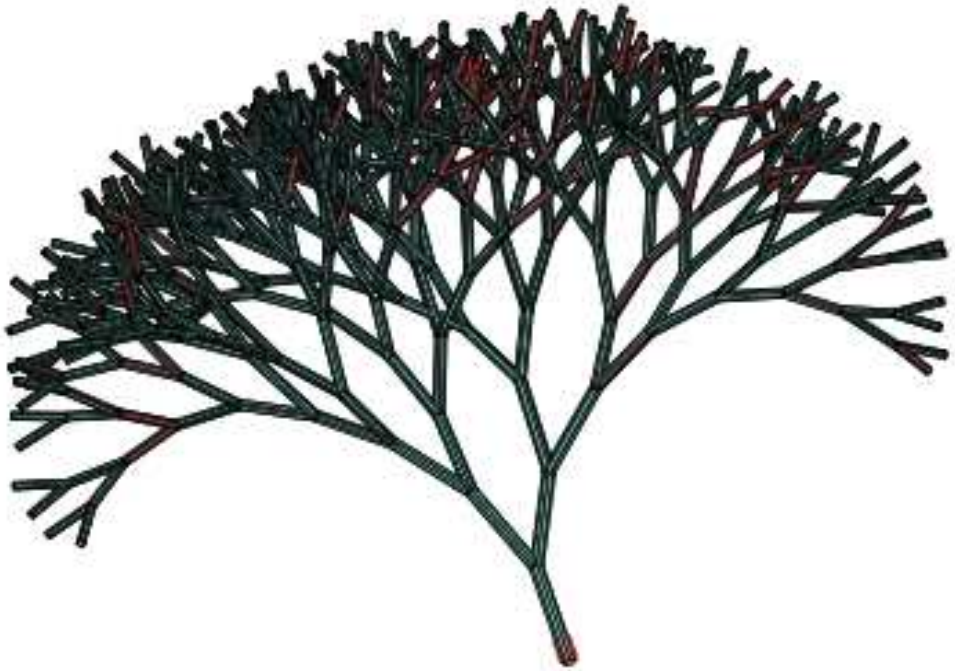
Lipowsky & Zweifach (1974) *Microvascular Res.*, **7**, 73-83.

Schmid-Schönbein *et al.* (1990) *Microvascular Res.*, **19**, 18-44.

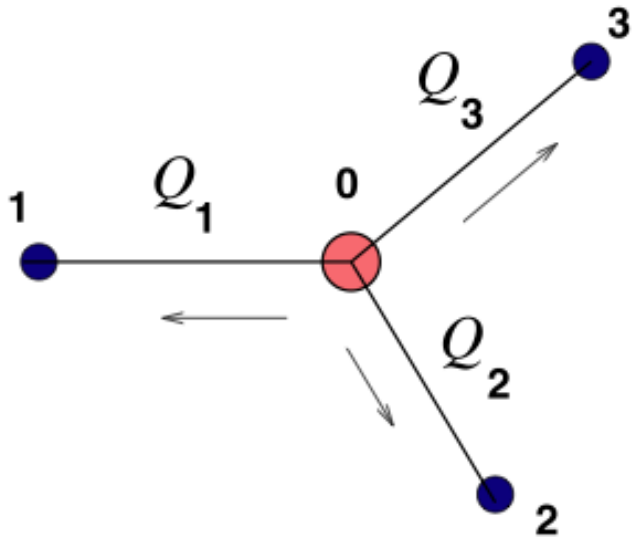
Pozrikidis (2009) *Bulletin Math. Biology*, **71**, 1520-1541.

Network models

Pozrikidis (2009)



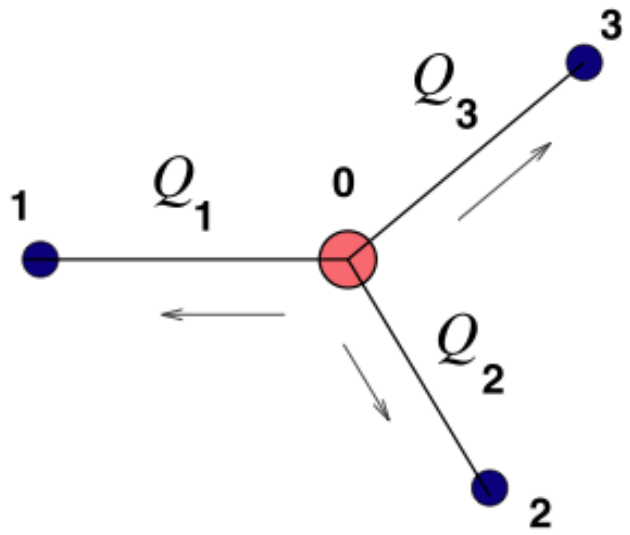
Network models: Pozrikidis (2009)



$$Q_i = \frac{\Delta p_i \, \pi a^4}{8 \mu L_i}$$

μ: Effective blood viscosity.

Network models: Pozrikidis (2009)



$P(\text{cell enters daughter branch } i) = \phi_i$

$$\phi_2 + \phi_3 = 1$$

Conclusion

Probabilities ϕ_i have important effect on cell residence times and haematocrit distribution across network.

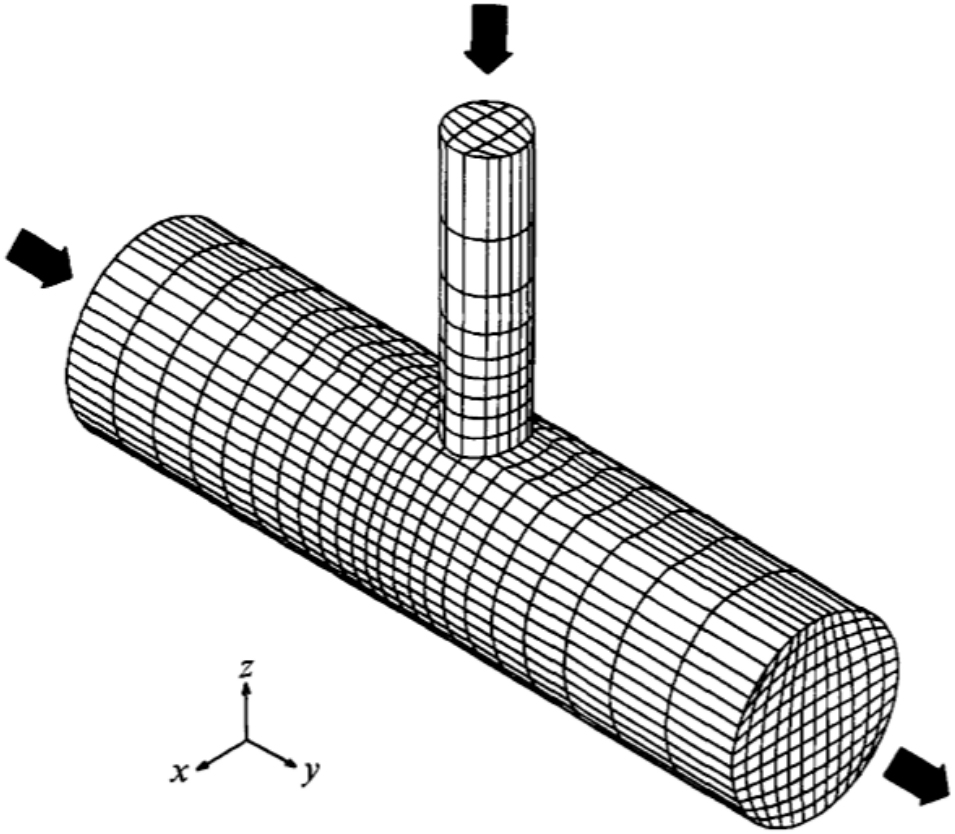
Flow simulations

Ong & Popel (1992) *J. Biomech. Eng.*, **114**, 398-405.

Ong, Enden & Popel (1994) *J. Fluid Mech.*, **270**

Focus on computing dividing fluid surface (no cells)

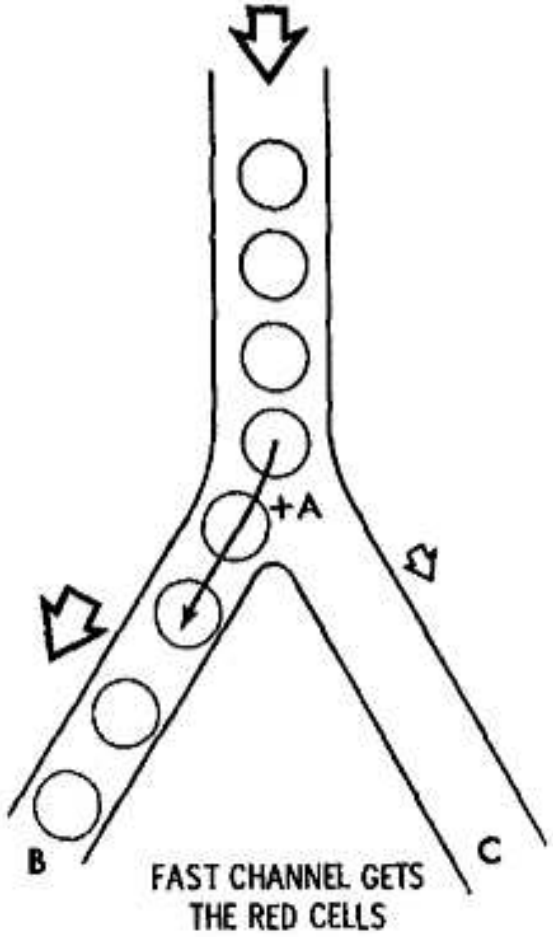
Finite-element calculations



Particle motion at a bifurcation

Fung (1973) *Microvascular Res.*, **5**, 34-48.

Most cells enter branch with higher flow rate

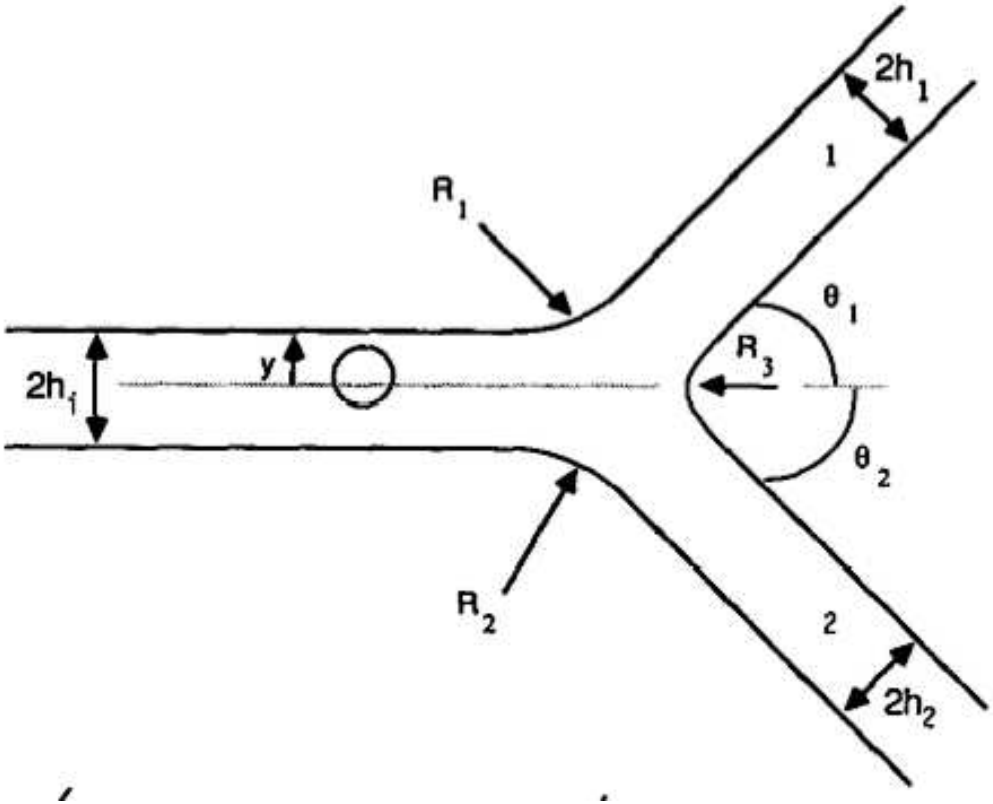


Particle motion at a bifurcation

Audet & Olbricht (1987) *Microvascular Res.*, **33**, 377-396.

Solid particle

Streamfunction-vorticity boundary integral approach

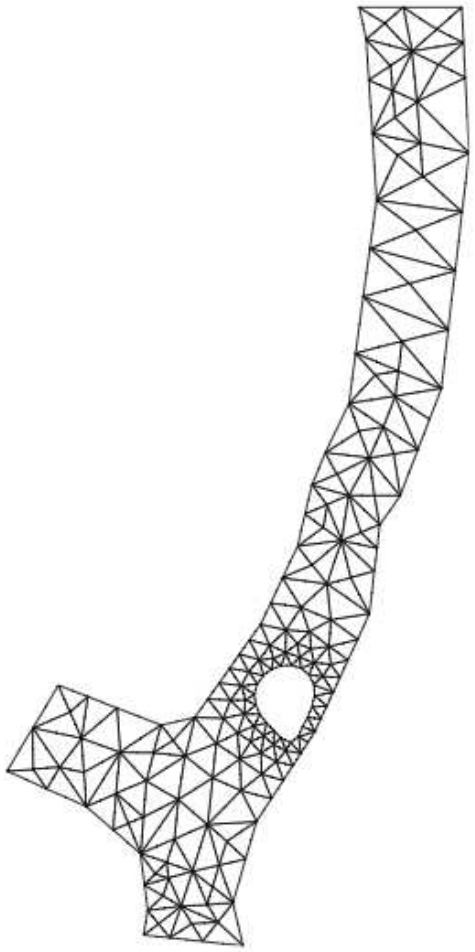


Cell motion at a bifurcation

Secomb *et al.* (2007) *Annals Biomed. Eng.*, **35**, 755-765.

Viscoelastic capsule motion

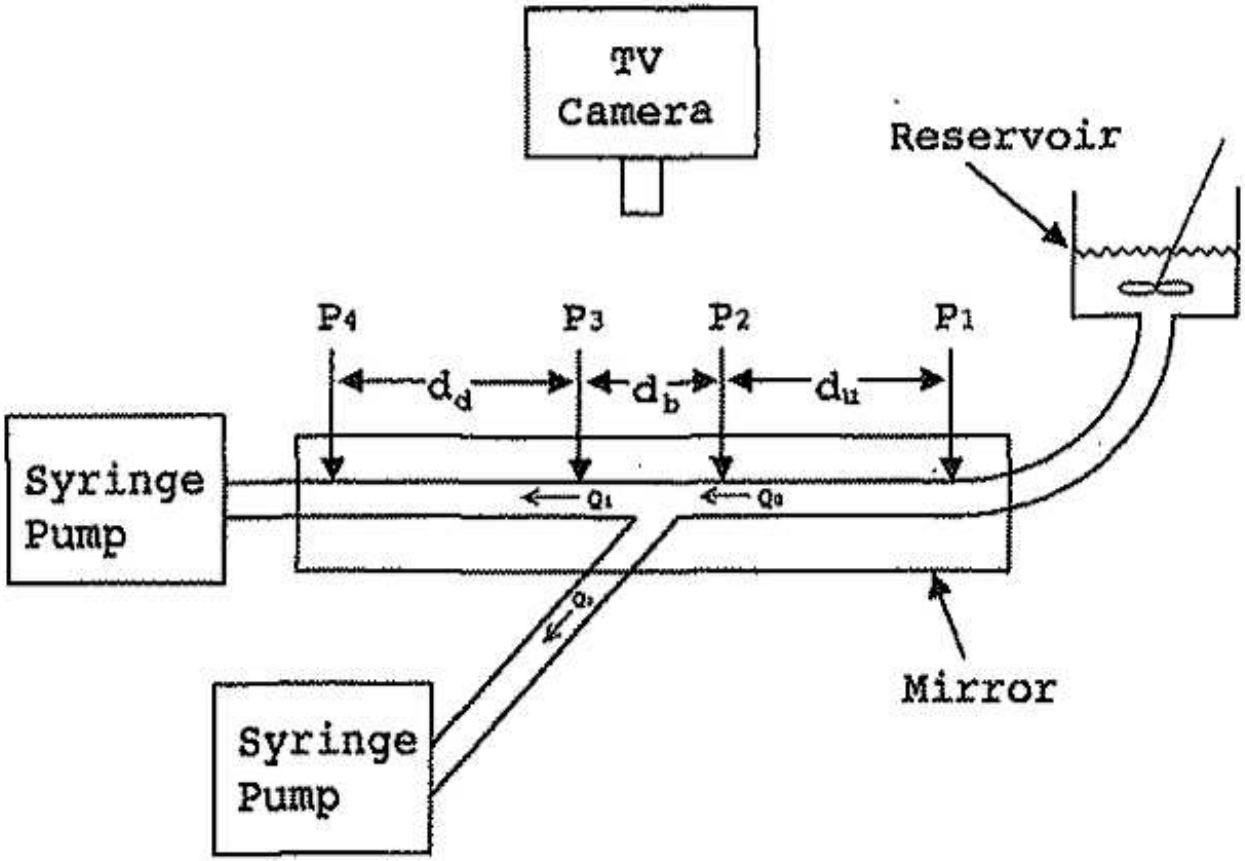
Finite-element method



Experiments

Kiani & Cokelet (2007) *J. Biomed. Eng.*, **116**, 497-501.

RBCs simulated by flexible elastic disks



Outline

The Boundary Integral Method

The boundary integral method stems from the Lorentz reciprocal relation (1907) for Stokes flow

$$\nabla \cdot \left(\mathbf{u} \cdot \boldsymbol{\sigma}' - \mathbf{u}' \cdot \boldsymbol{\sigma} \right) = \mathbf{0},$$

for two distinct flows $(\mathbf{u}, \boldsymbol{\sigma})$ and $(\mathbf{u}', \boldsymbol{\sigma}')$ which satsify the Stokes equations:

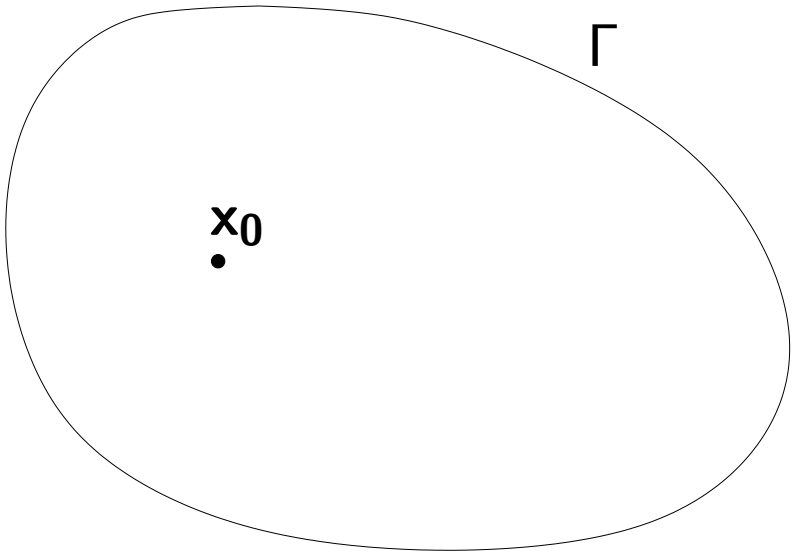
$$\begin{aligned} \mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u} &= \nabla \cdot \boldsymbol{\sigma}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \qquad \begin{aligned} \delta(\mathbf{x} - \mathbf{x}_0) &= \nabla \cdot \boldsymbol{\sigma}' \\ \nabla \cdot \mathbf{u'} &= 0 \end{aligned}$$

The boundary integral method

Integrate

$$\nabla \cdot \left(\mathbf{u} \cdot \boldsymbol{\sigma}' - \mathbf{u}' \cdot \boldsymbol{\sigma} \right) = \mathbf{0}$$

over Γ.



The boundary integral method

We find

$$u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_\Gamma f_i(\mathbf{x})G_{ij}(\mathbf{x},\mathbf{x}_0)\mathrm{d}s(\mathbf{x}) + \frac{1}{4\pi}\int_\Gamma u_i(\mathbf{x})T_{ijk}(\mathbf{x},\mathbf{x}_0)n_k\mathrm{d}s(\mathbf{x}),$$

where $f_i = \sigma_{ij}n_j$ is the traction on the boundary Γ.

The boundary integral method

We find

$$u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_\Gamma f_i(\mathbf{x})\,G_{ij}(\mathbf{x},\mathbf{x}_0)\mathrm{d}s(\mathbf{x}) + \frac{1}{4\pi}\int_\Gamma u_i(\mathbf{x})\,T_{ijk}(\mathbf{x},\mathbf{x}_0)n_k\mathrm{d}s(\mathbf{x}),$$

where $f_i = \sigma_{ij}n_j$ is the traction on the boundary Γ.

The Green's function G_{ij} is a solution $u_i = G_{ij}b_j$ of

$$\mu\nabla^2\mathbf{u} + \nabla p + \mathbf{b}\,\delta(\mathbf{x}-\mathbf{x}_0) = \mathbf{0}$$

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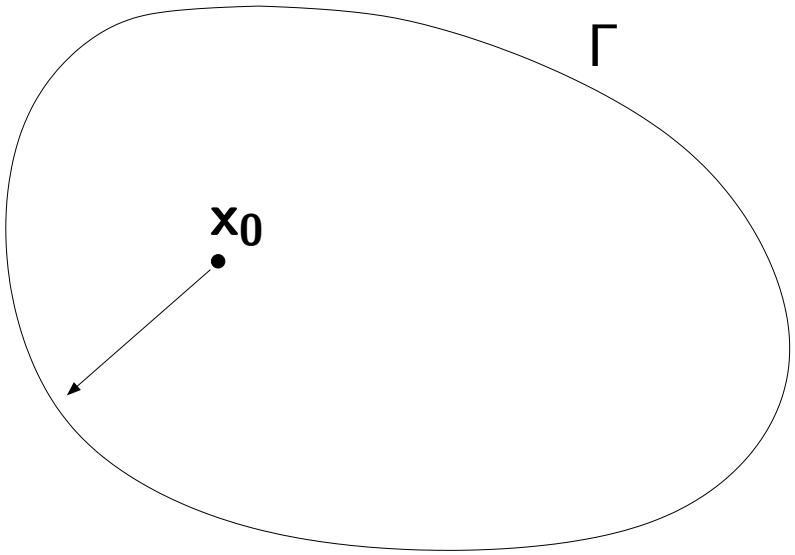
Free-space Green's function:

$$G_{ij}(\mathbf{x},\mathbf{x}_0) = -\delta_{ij}\log r + \frac{\hat{x}_i\hat{x}_j}{r^2}, \quad \hat{\mathbf{x}} = \mathbf{x}-\mathbf{x}_0, \quad r = |\hat{\mathbf{x}}|.$$

The boundary integral method

Let $\mathbf{x}_0$ approach the boundary.



The boundary integral method

Now

$$u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_\Gamma f_i(\mathbf{x})\, G_{ij}(\mathbf{x},\mathbf{x}_0)\ \mathrm{d}s + \frac{1}{4\pi}\int_\Gamma u_i(\mathbf{x})\, T_{ijk}(\mathbf{x},\mathbf{x}_0) n_k\ \mathrm{d}s,$$

with $\mathbf{x}_0$ inside Γ

becomes (Ladyzhenskaya 1963)

$$\frac{1}{2}u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_\Gamma f_i(\mathbf{x}) G_{ij}(\mathbf{x},\mathbf{x}_0)\mathrm{d}s + \frac{1}{4\pi}\int_\Gamma^{PV} u_i(\mathbf{x})\, T_{ijk}(\mathbf{x},\mathbf{x}_0) n_k \mathrm{d}s.$$

with $\mathbf{x}_0$ on Γ

Outline

Ultimate aim

To model the motion of red blood cells through a capillary bifurcation.

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Present simplifications

Two-dimensional model

Single cell moving through a tube with a side branch

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To model the motion of red blood cells through a capillary bifurcation.

Present simplifications

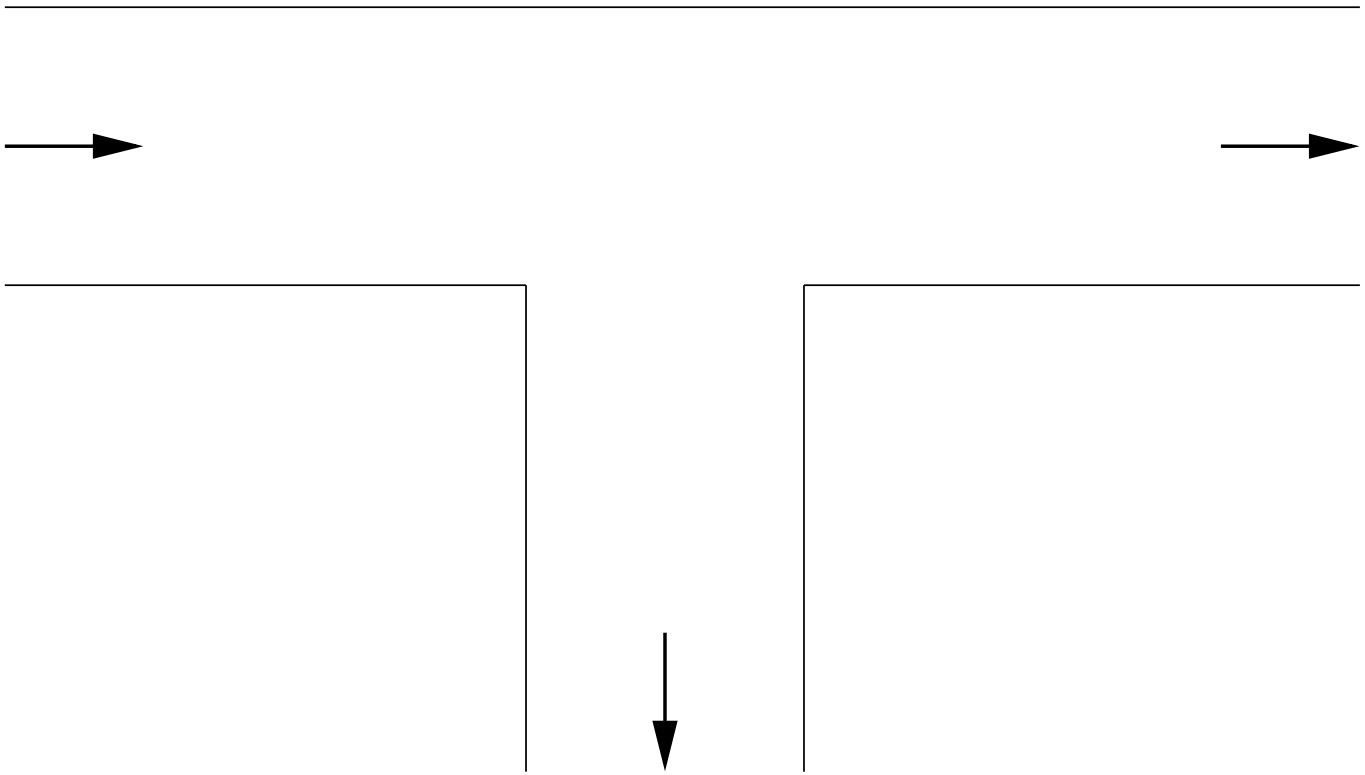
Two-dimensional model

Single cell moving through a tube with a side branch

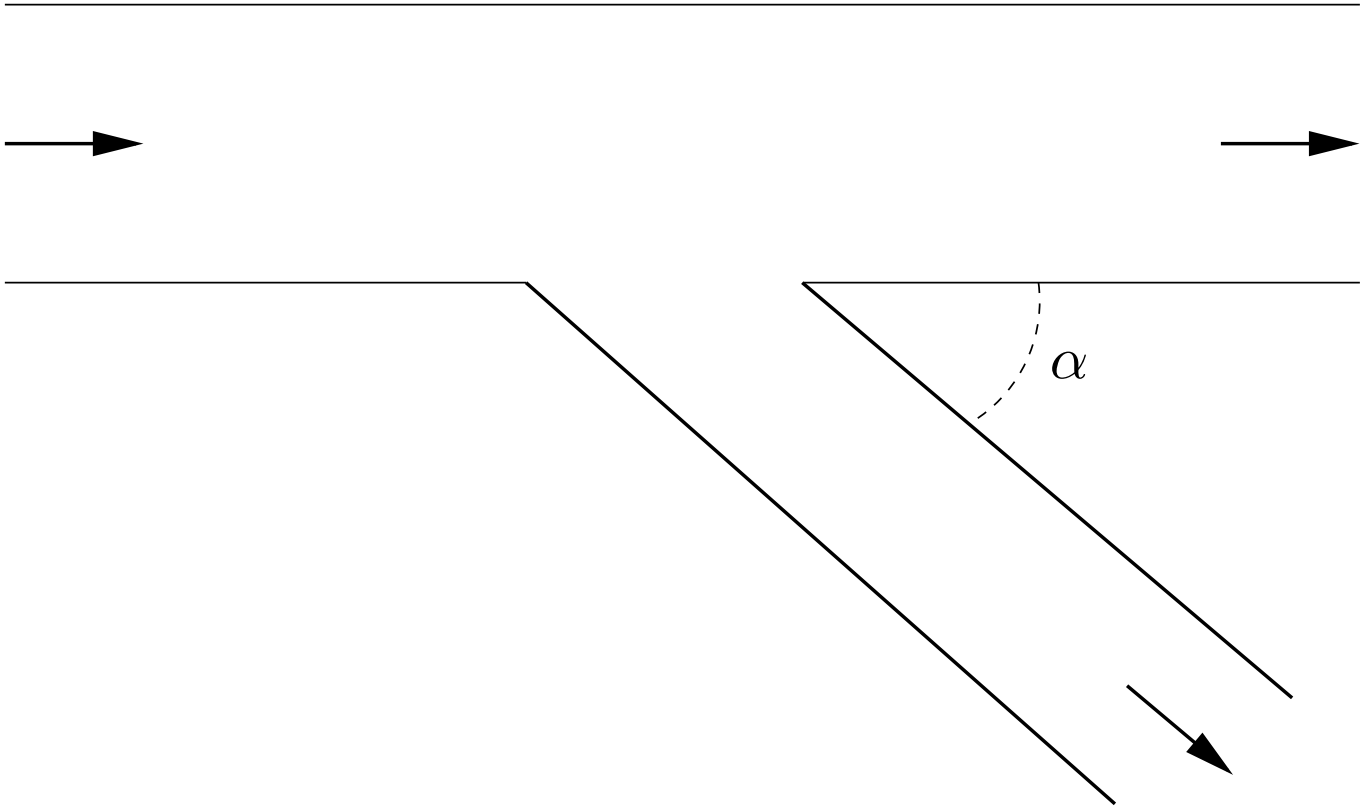
3D branching flow

Work on this is at an early stage

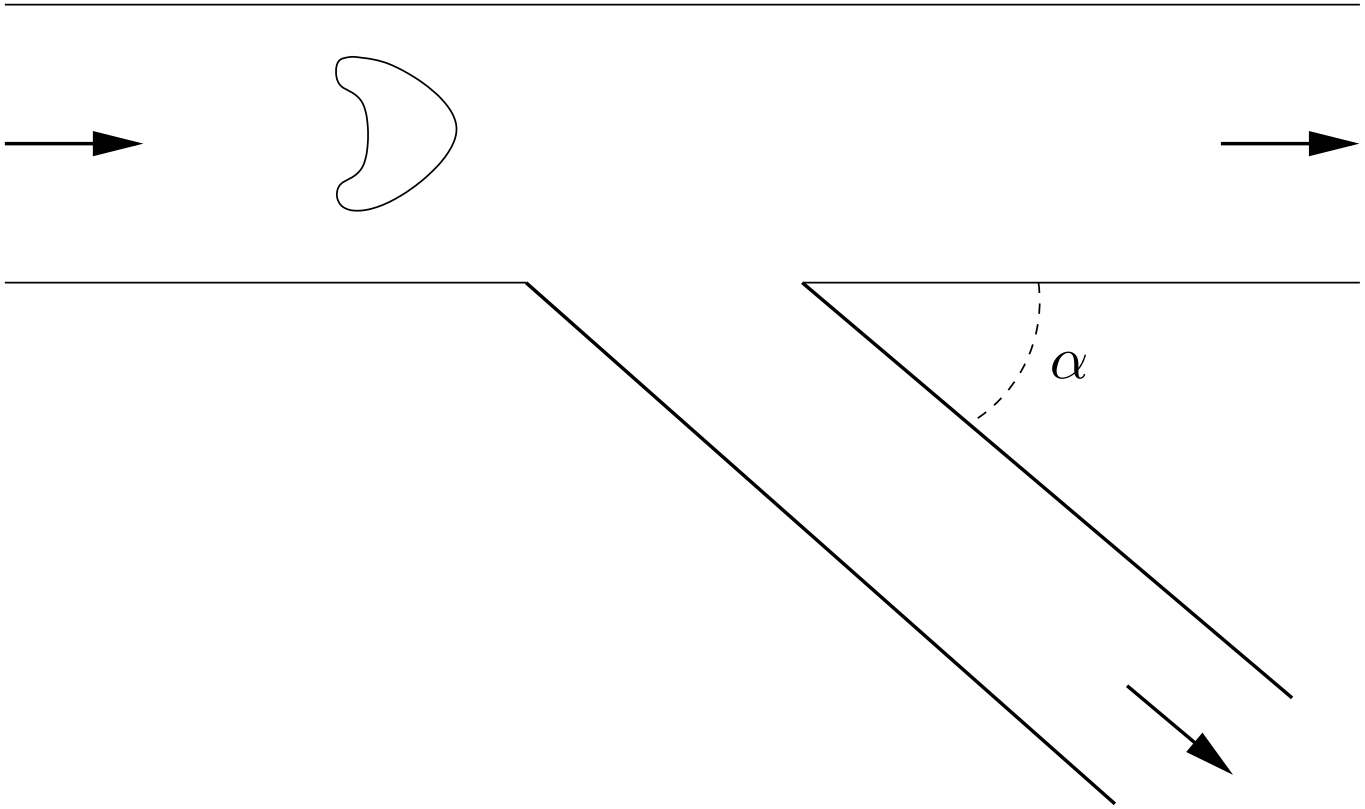
Model geometry



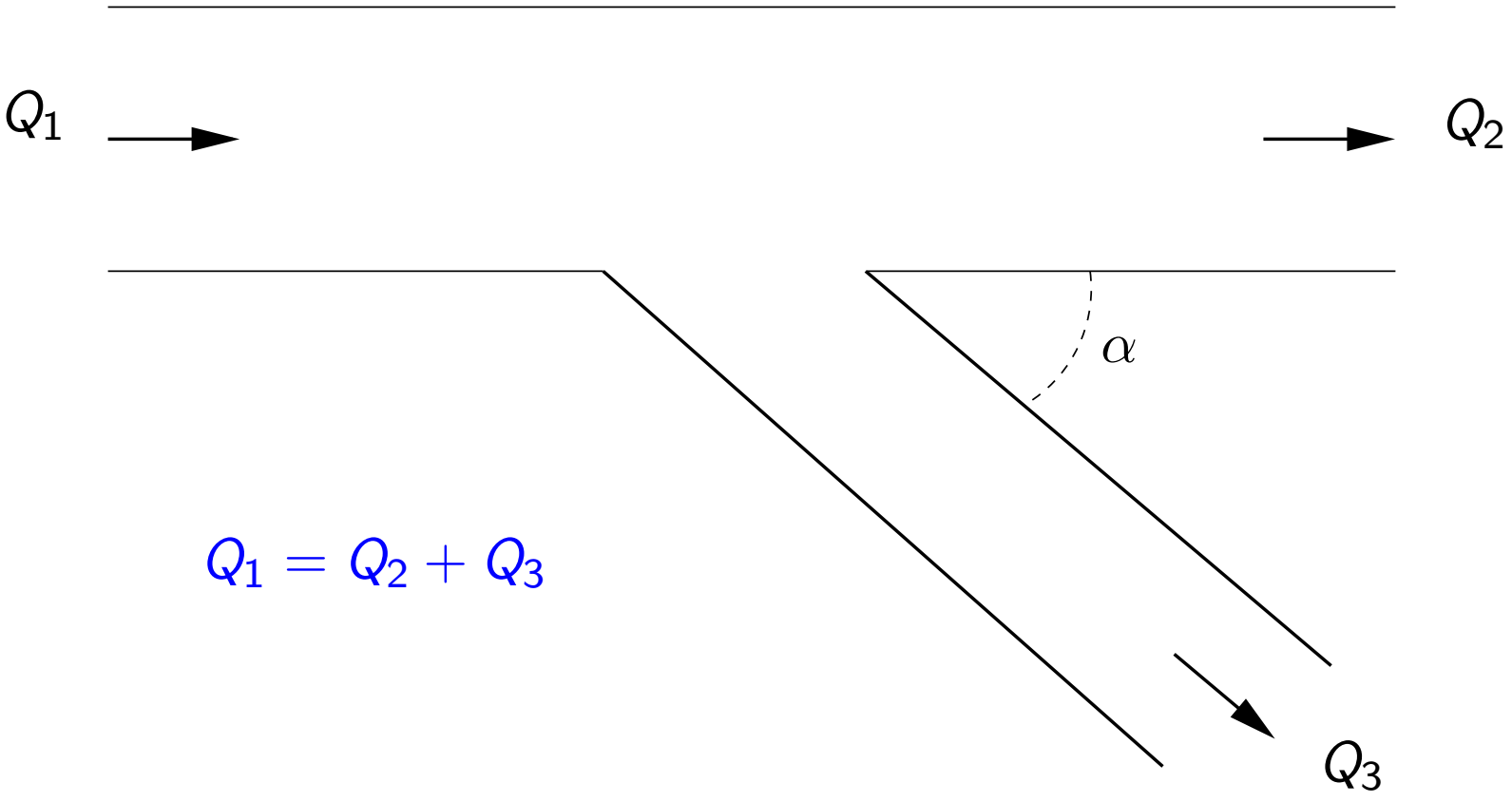
Model geometry



Model geometry

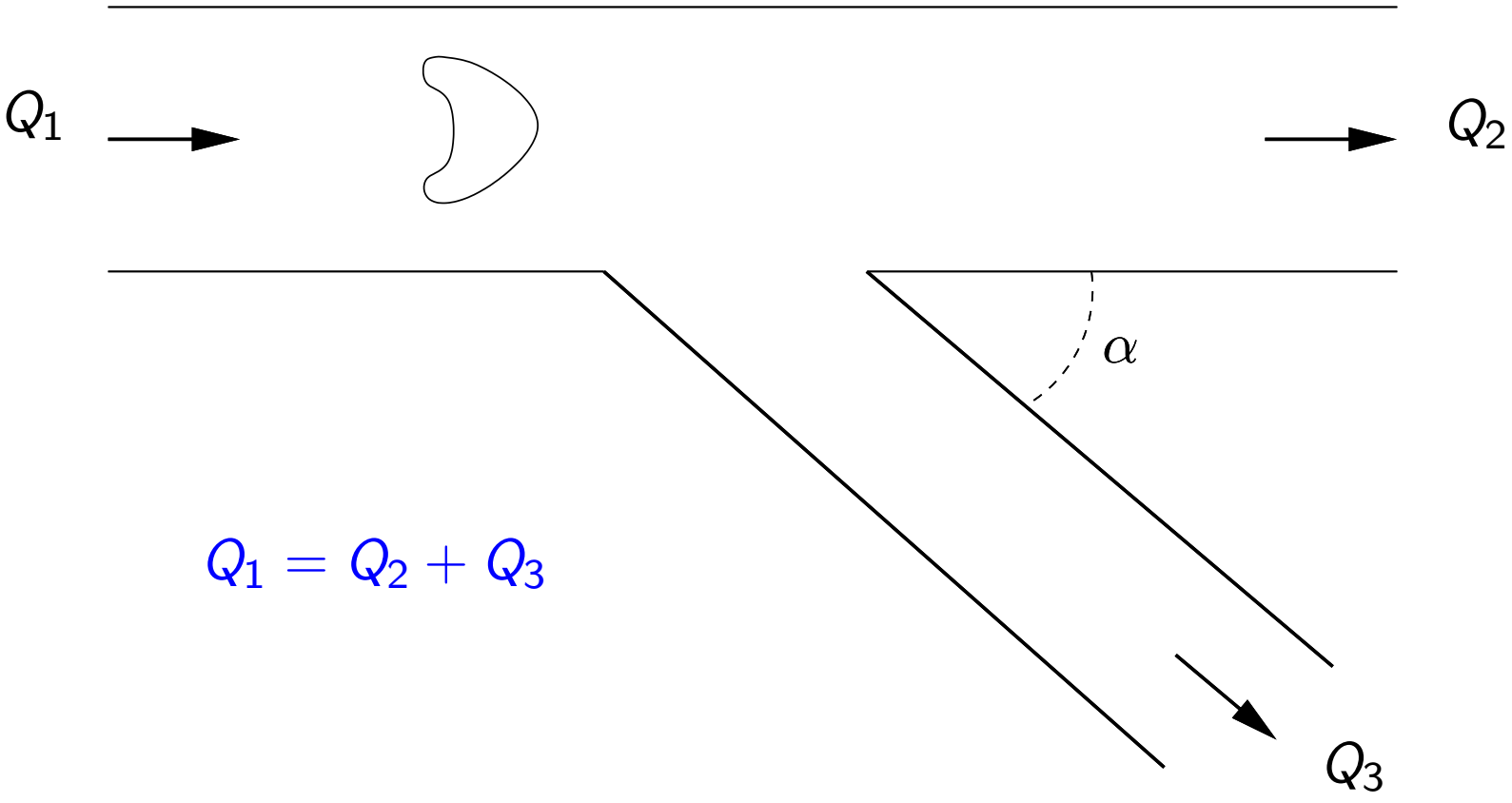


Flow set-up



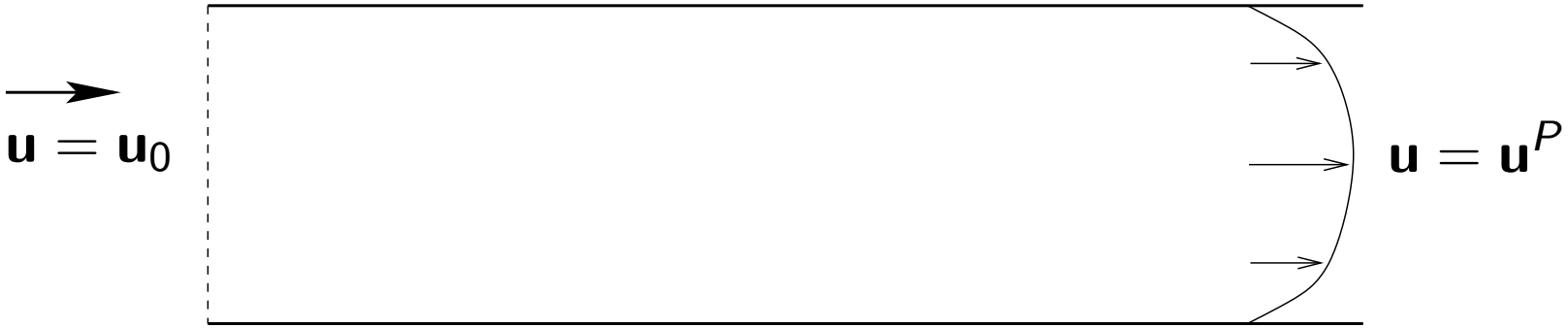
$$Q_1 = Q_2 + Q_3$$

Flow set-up

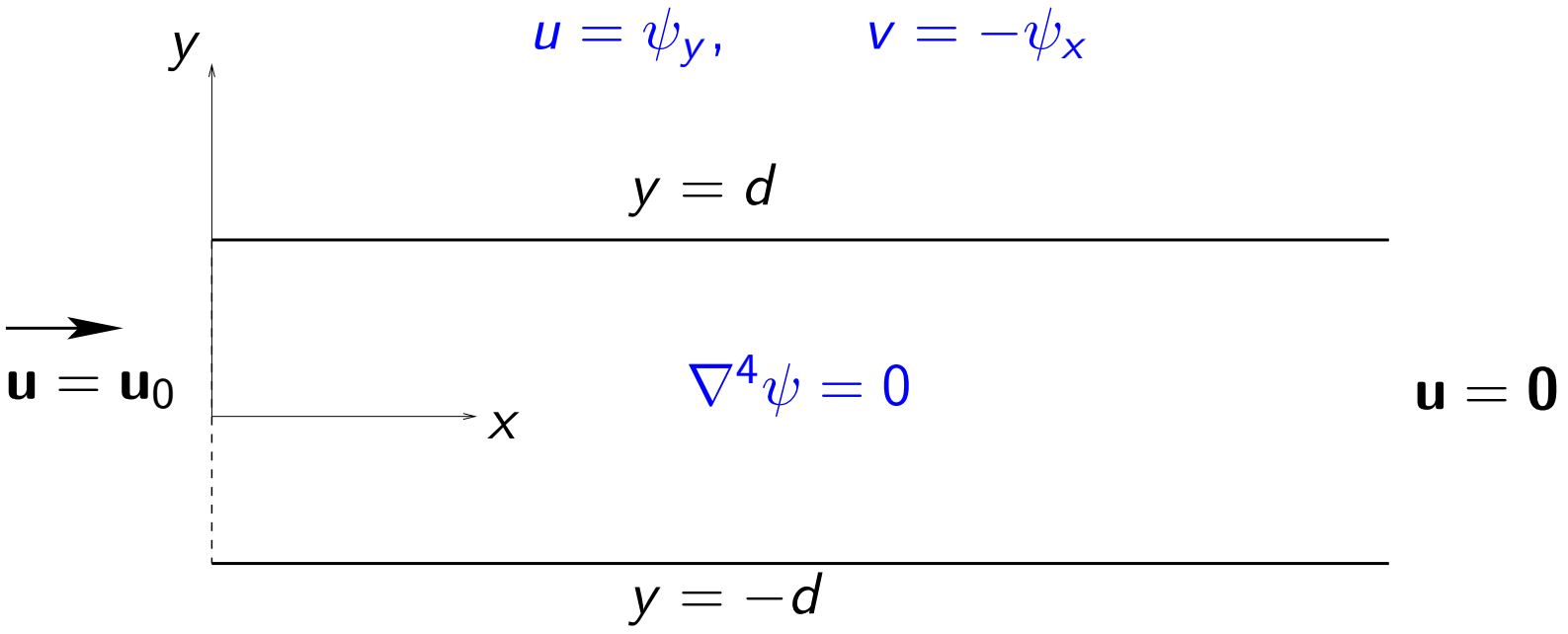


$$Q_1 = Q_2 + Q_3$$

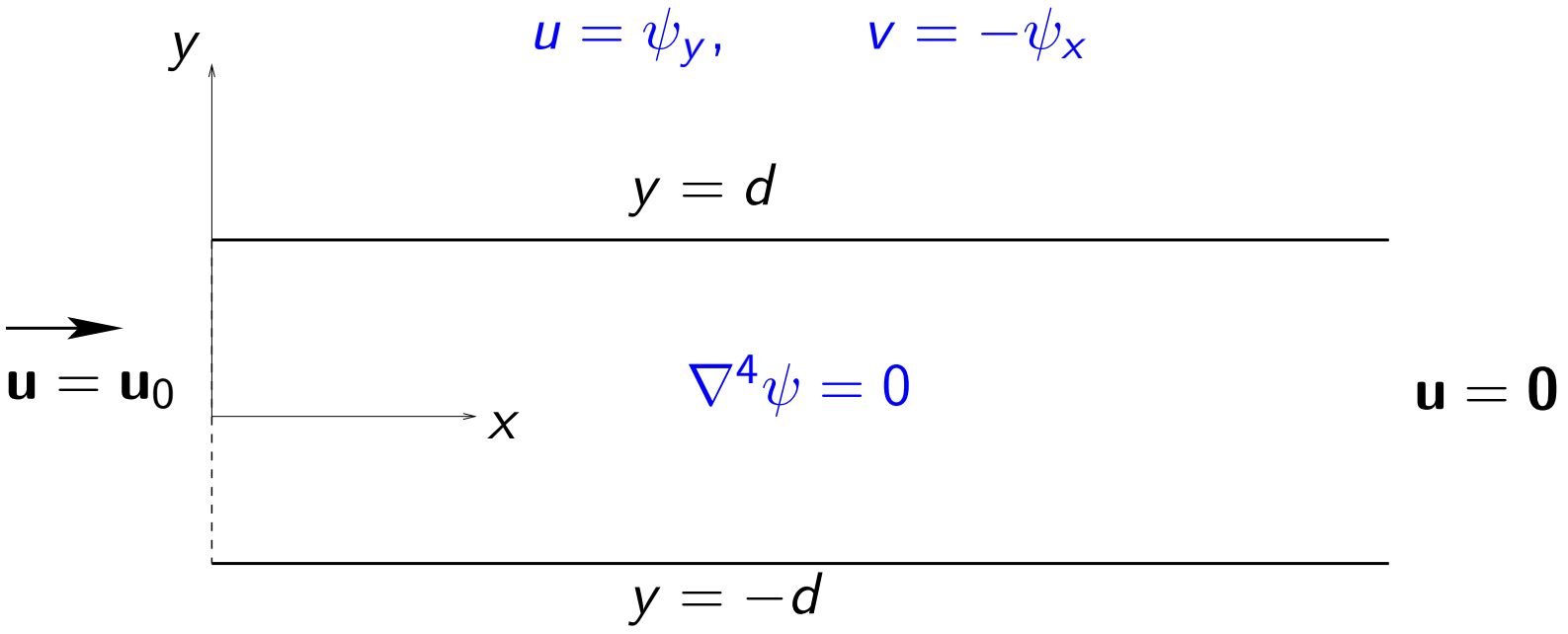
Decay rates in Stokes flow



Decay rates in Stokes flow

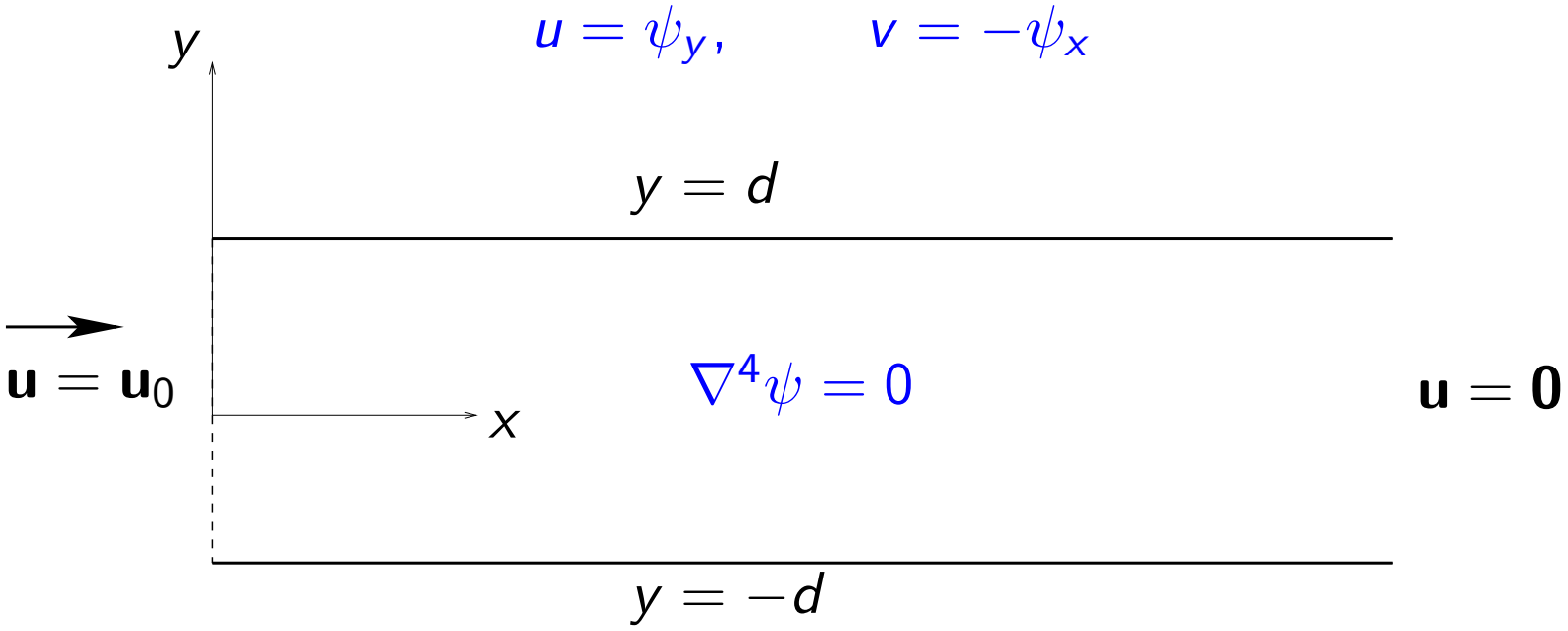


Decay rates in Stokes flow



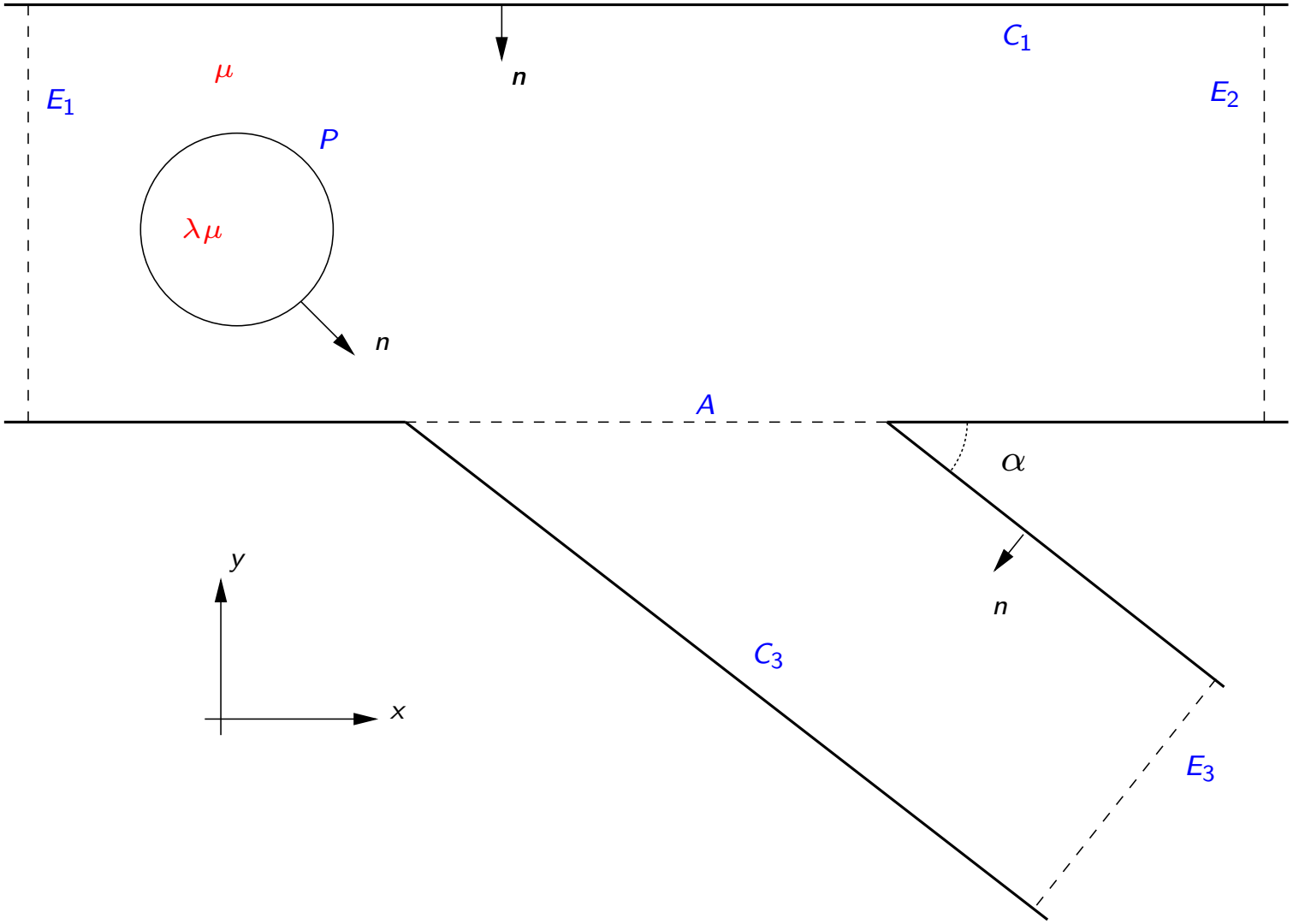
$$\psi = \sum_{n=0}^{\infty} \alpha_n \, \phi_n(y) \, \mathrm{e}^{-k_n x}$$

Decay rates in Stokes flow

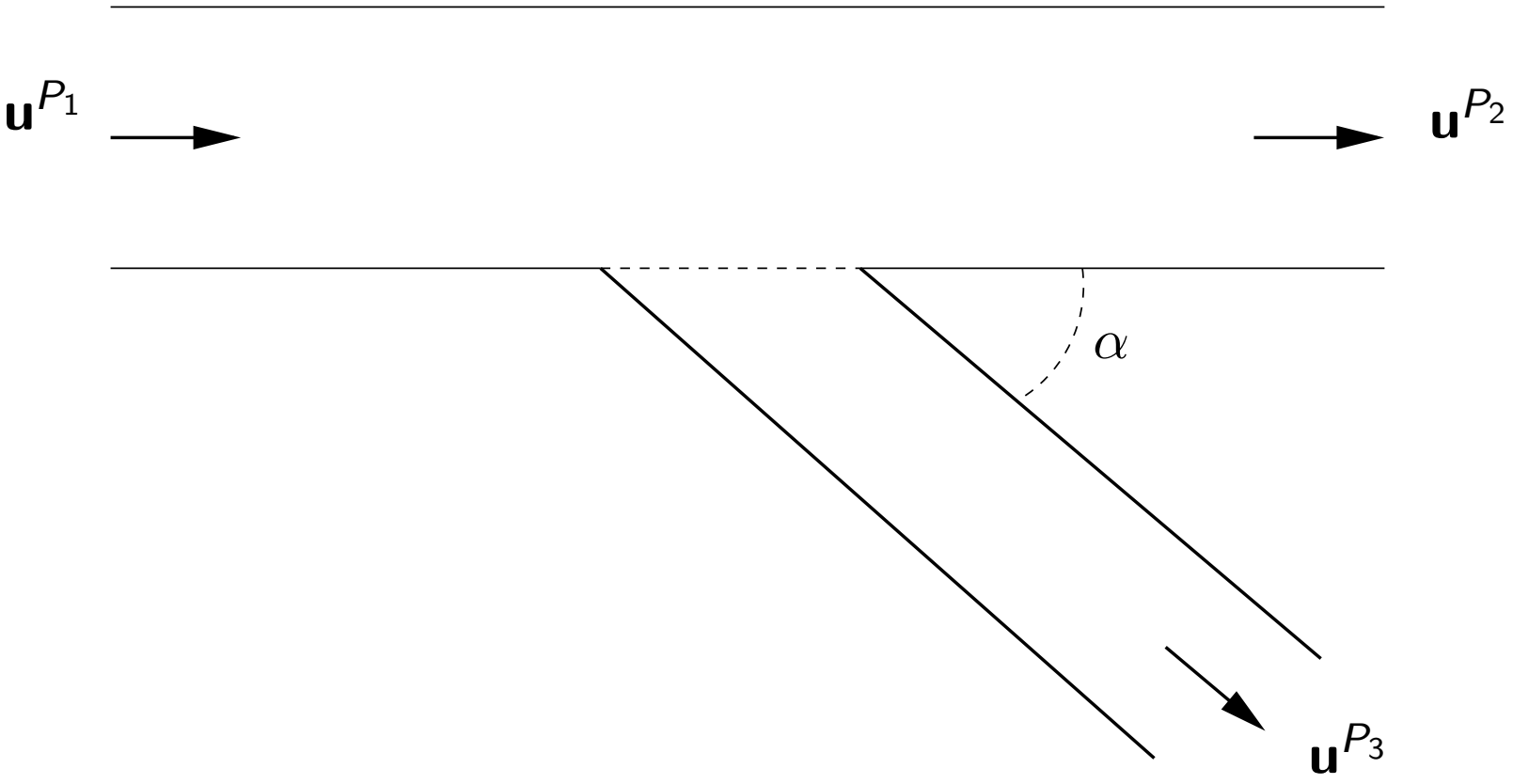


$$k_0 = \frac{2.11}{d} + \frac{1.125\,\mathrm{i}}{d}$$

Computational model



Flow set-up



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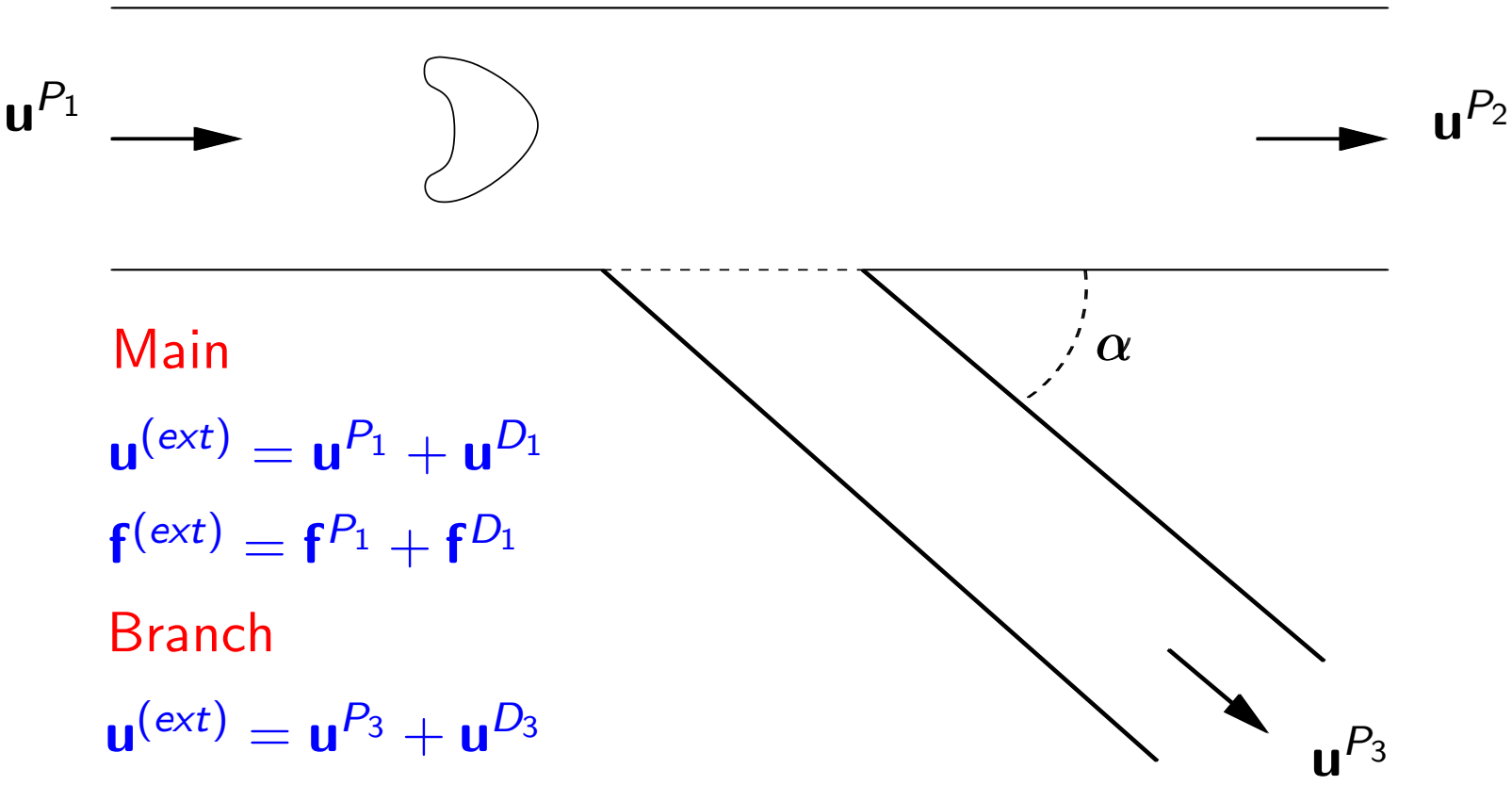


Main

$$\mathbf{u}^{(ext)} = \mathbf{u}^{P_1} + \mathbf{u}^{D_1}$$

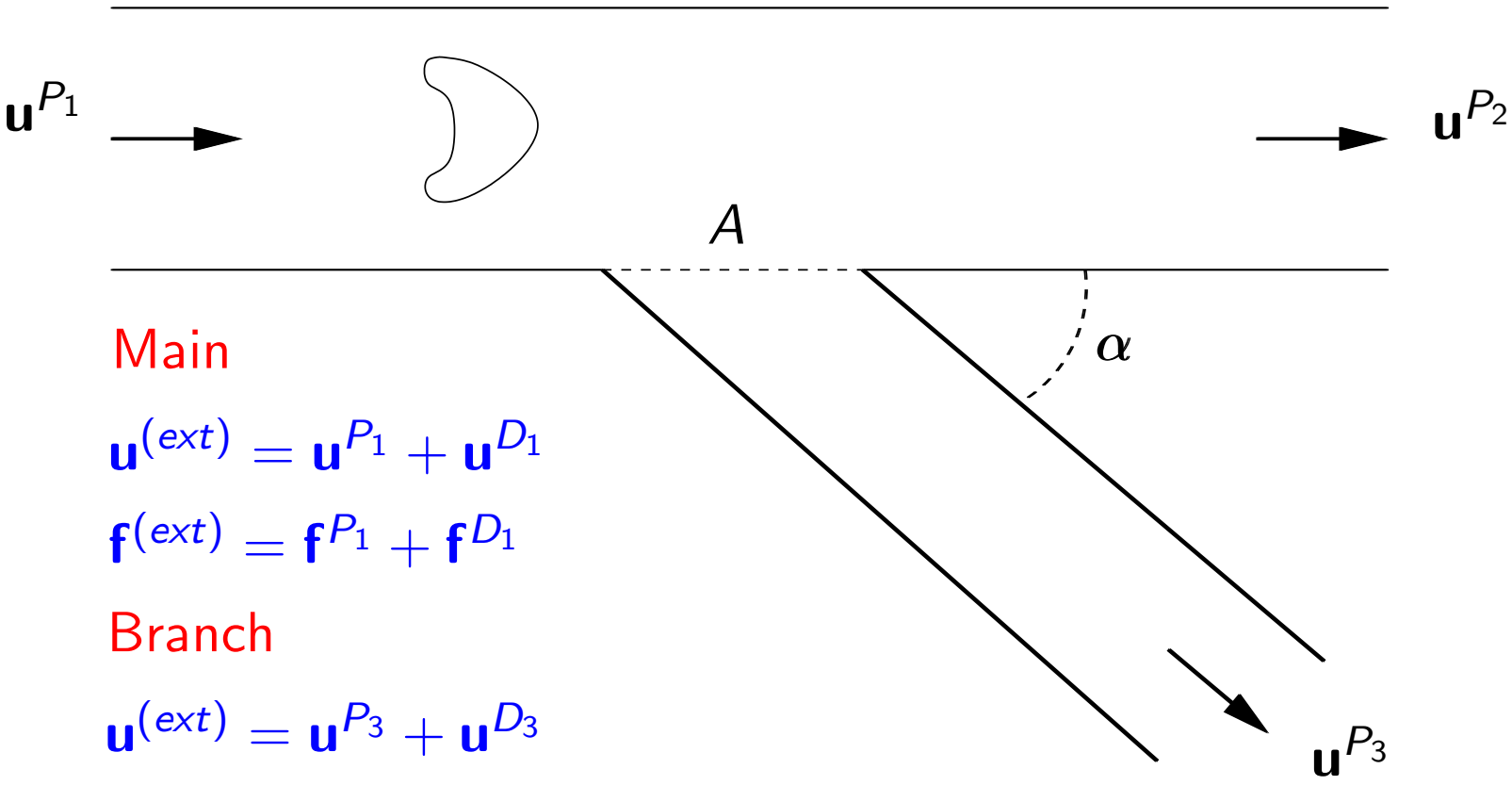
$$\mathbf{f}^{(ext)} = \mathbf{f}^{P_1} + \mathbf{f}^{D_1}$$

Branch

$$\mathbf{u}^{(ext)} = \mathbf{u}^{P_3} + \mathbf{u}^{D_3}$$

$$\mathbf{f}^{(ext)} = \mathbf{f}^{P_3} + \mathbf{f}^{D_3}$$

Flow set-up



Main

$$\mathbf{u}^{(ext)} = \mathbf{u}^{P_1} + \mathbf{u}^{D_1}$$

$$\mathbf{f}^{(ext)} = \mathbf{f}^{P_1} + \mathbf{f}^{D_1}$$

Branch

$$\mathbf{u}^{(ext)} = \mathbf{u}^{P_3} + \mathbf{u}^{D_3}$$

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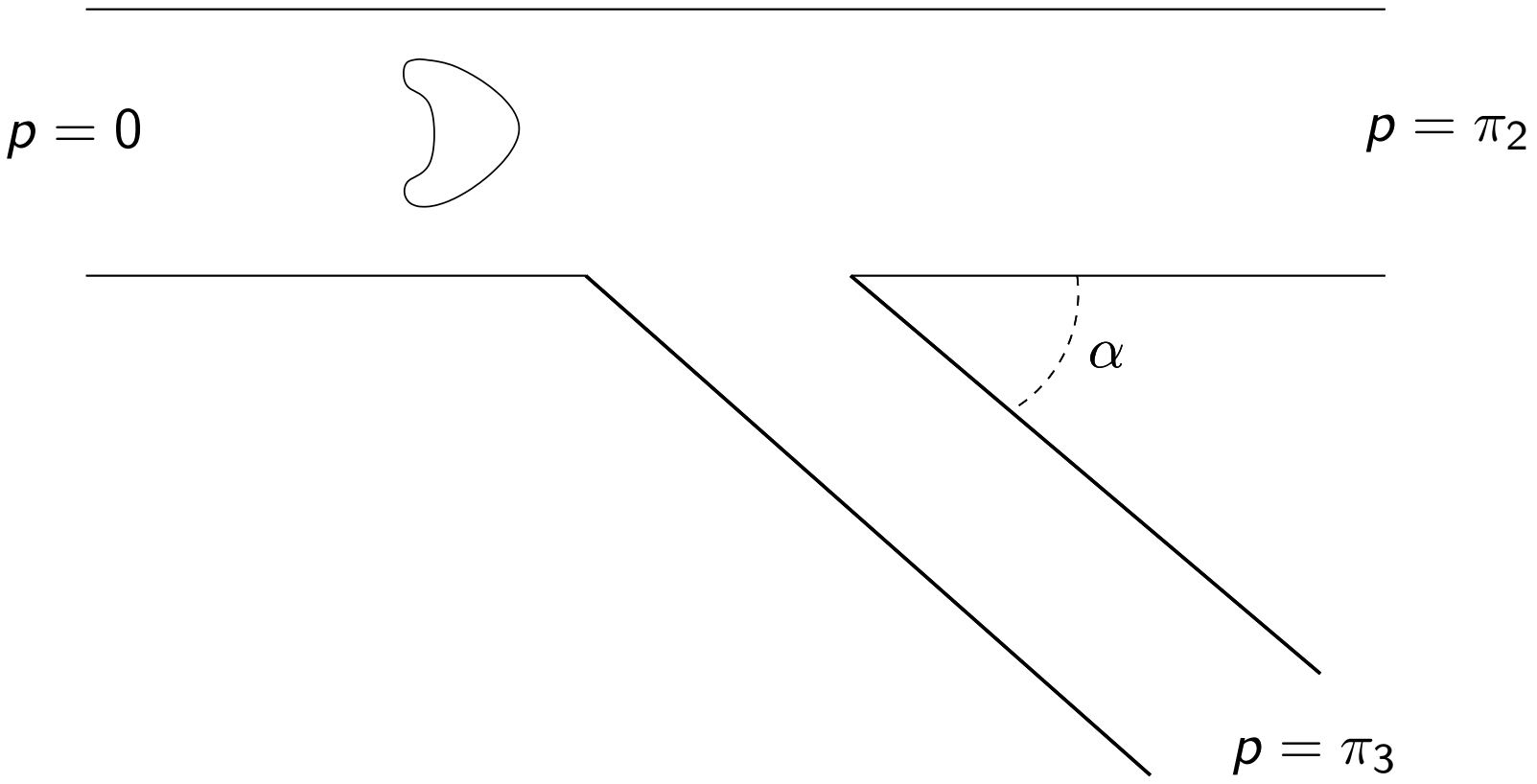
Flow set-up



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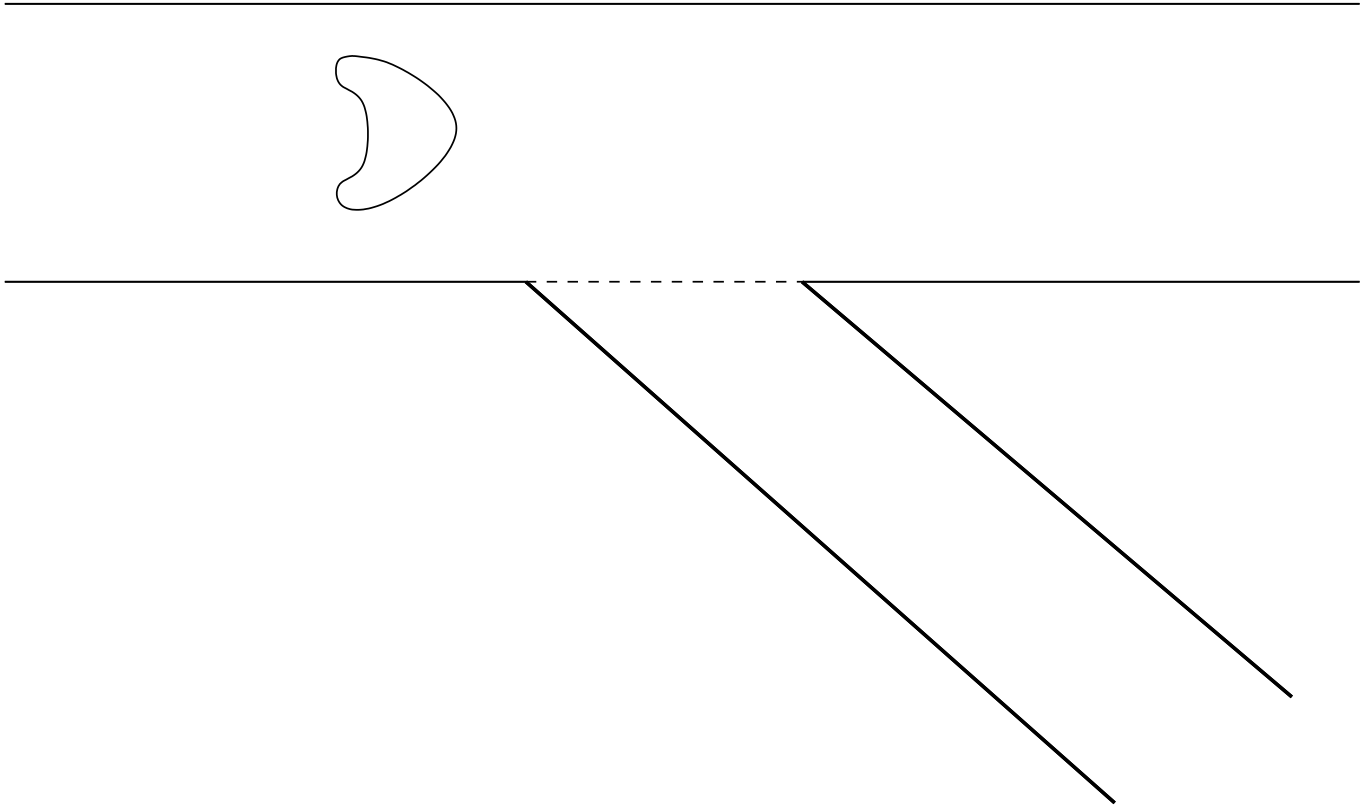

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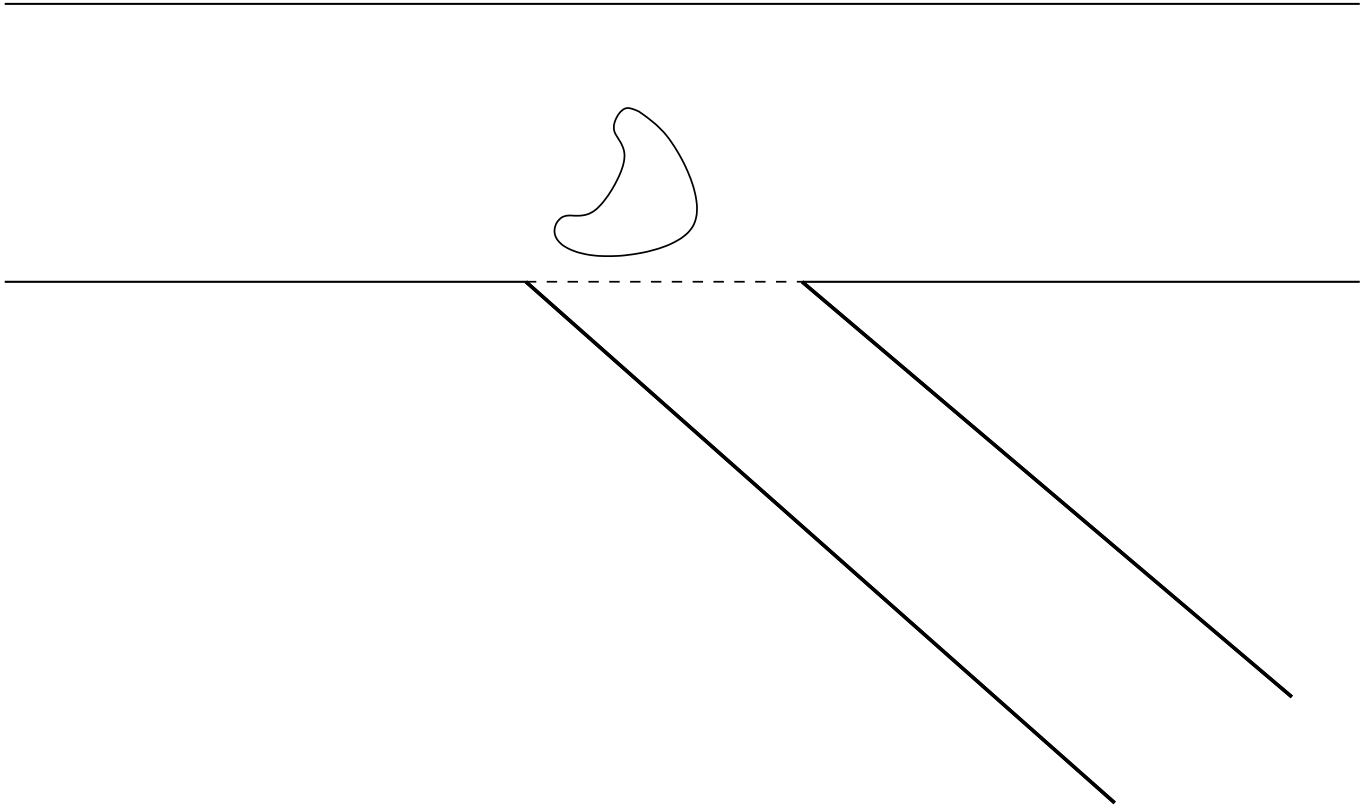

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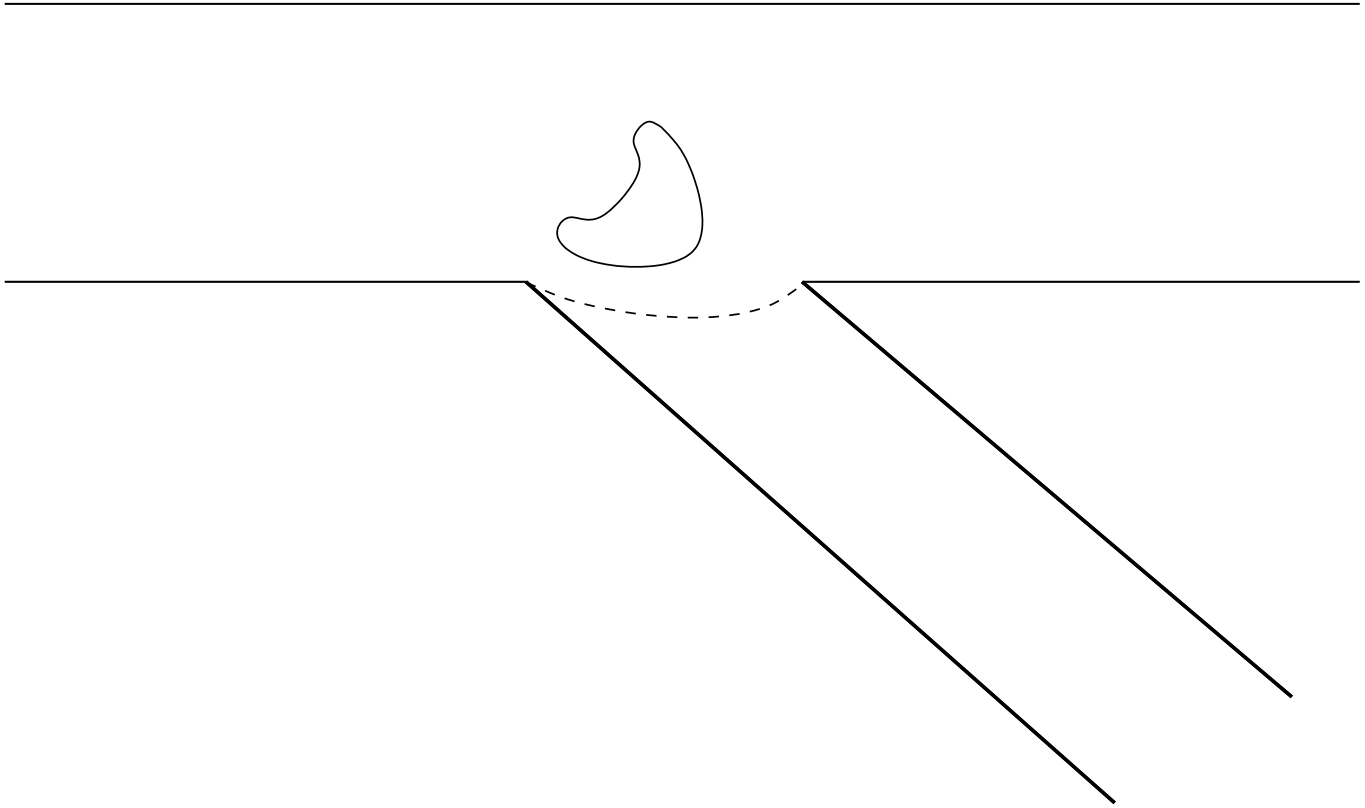

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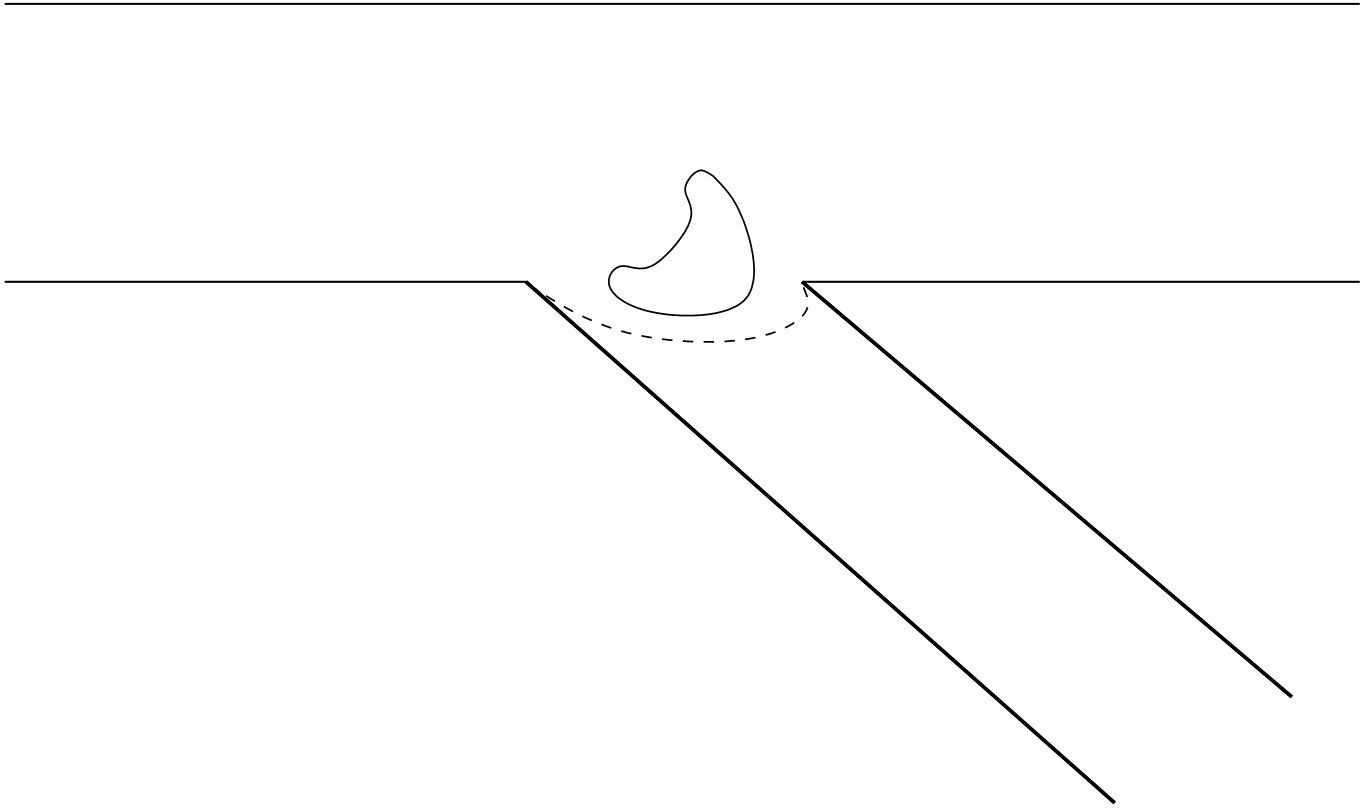

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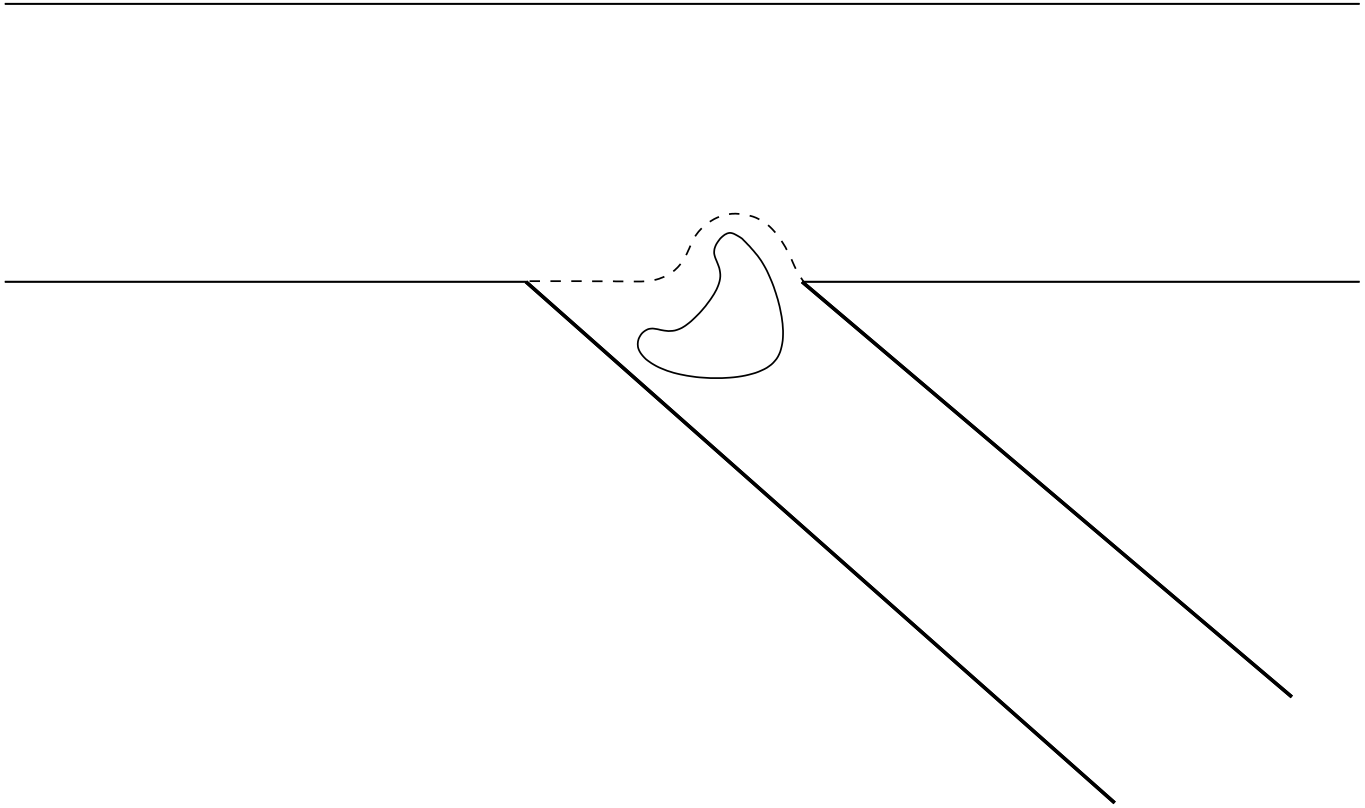

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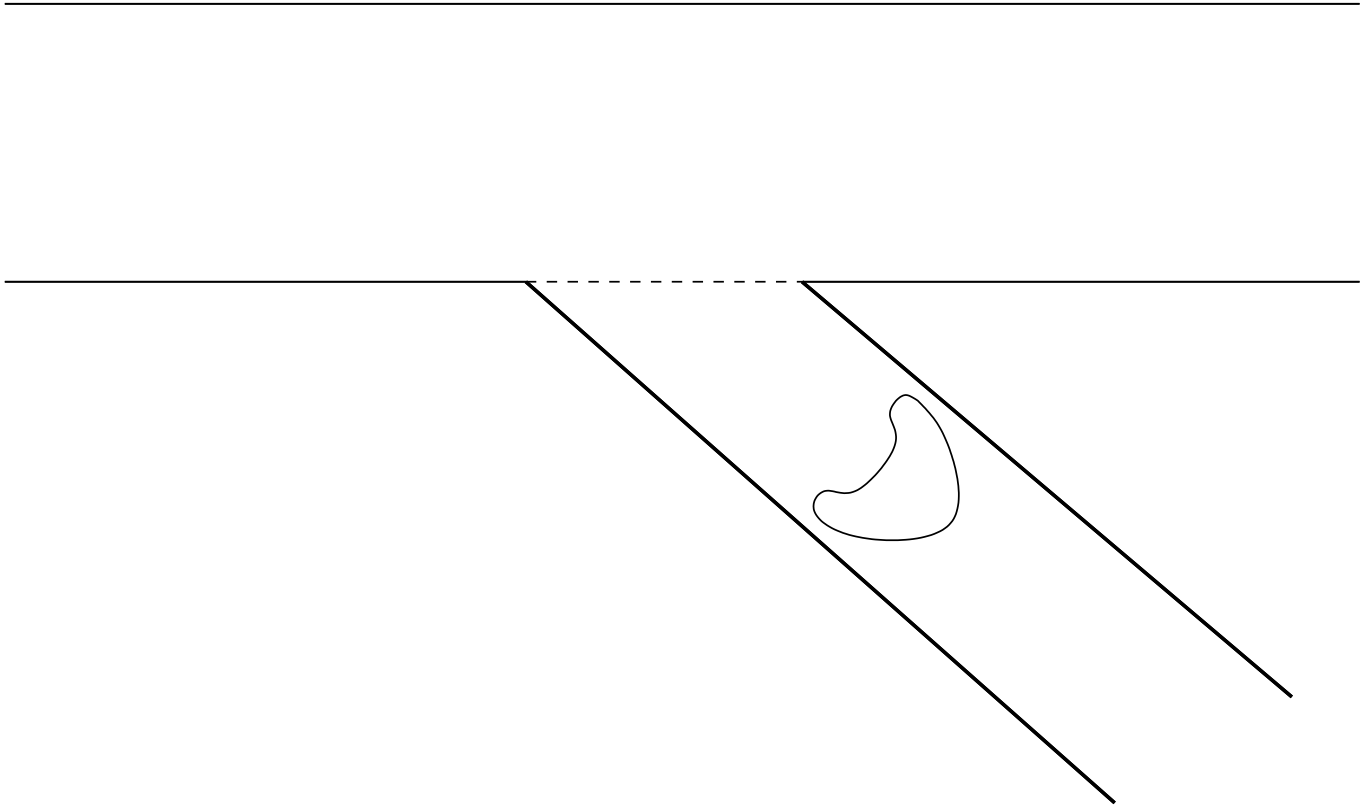


Boundary integral formulation

We apply the standard boundary integral formulation

$$u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_\Gamma f_i(\mathbf{x})G_{ij}(\mathbf{x},\mathbf{x}_0)\mathrm{d}s(\mathbf{x}) + \frac{1}{4\pi}\int_\Gamma u_i(\mathbf{x})T_{ijk}(\mathbf{x},\mathbf{x}_0)n_k\mathrm{d}s(\mathbf{x}),$$

to the disturbance flows.

Boundary integral formulation

We find

$$2\pi\mu\, u_j^{D_1}(\mathbf{x_0}) = -\int_{C,A} f_i^{D_1}(\mathbf{x})\, G_{ij}(\mathbf{x},\mathbf{x_0})\, \mathrm{d}s(\mathbf{x}) + \mu \int_A u_i^{D_1}(\mathbf{x})\, T_{ijk}(\mathbf{x},\mathbf{x_0})\, n_k\, \mathrm{d}s(\mathbf{x})$$
$$+ (\pi_2 + Q\Delta p) \int_{E_2} G_{ij}(\mathbf{x},\mathbf{x_0})\, n_i(\mathbf{x})\, \mathrm{d}s(\mathbf{x}) + I_j(\mathbf{x_0}) - \int_P \Delta f_i(\mathbf{x})\, G_{ij}(\mathbf{x},\mathbf{x_0})\, \mathrm{d}s(\mathbf{x})$$
$$+ \mu(1-\lambda) \int_P u_i^{(ext)}(\mathbf{x})\, T_{ijk}(\mathbf{x},\mathbf{x_0})\, n_k(\mathbf{x})\, \mathrm{d}s(\mathbf{x}),$$

$$I_j(\mathbf{x_0}) = (1-Q)\left[\int_{E_2} f_i^{P_1}(\mathbf{x}) G_{ij}(\mathbf{x},\mathbf{x_0})\, \mathrm{d}s(\mathbf{x}) - \mu \int_{E_2} u_i^{P_1}(\mathbf{x}) T_{ijk}(\mathbf{x},\mathbf{x_0})\, n_k(\mathbf{x})\, \mathrm{d}s(\mathbf{x})\right]$$

where

$$Q = \frac{Q_2}{Q_1}, \quad \text{and} \quad \Delta\mathbf{f} = \mathbf{f}^{(ext)} - \mathbf{f}^{(int)}$$

is the traction jump at the cell boundary.

Boundary integral formulation

In the branch we find

$$2\pi\mu\, u_j^{D_3}(\mathbf{x_0}) = \int_A f_i^{D_1}(\mathbf{x}) G_{ij}(\mathbf{x},\mathbf{x_0}) \mathrm{ds}(\mathbf{x}) - \int_B f_i^{D_3}(\mathbf{x}) G_{ij}(\mathbf{x},\mathbf{x_0}) \mathrm{ds}(\mathbf{x}) + K_j(\mathbf{x_0})$$
$$- \mu \int_A u_i^{D_1}(\mathbf{x}) T_{ijk}(\mathbf{x},\mathbf{x_0}) n_k(\mathbf{x})\, \mathrm{ds}(\mathbf{x}) + \pi_3 \int_{E_3} G_{ij}(\mathbf{x},\mathbf{x_0})\, n_i(\mathbf{x})\, \mathrm{ds}(\mathbf{x})$$

where

$$K_j(\mathbf{x_0}) = \int_A \left[f_i^{P_1}(\mathbf{x}) - f_i^{P_3}(\mathbf{x}) \right] G_{ij}\, \mathrm{ds}(\mathbf{x}) + \mu \int_A \left[u_i^{P_3}(\mathbf{x}) - u_i^{P_1}(\mathbf{x}) \right] T_{ijk}\, n_k(\mathbf{x}) \mathrm{ds}(\mathbf{x}).$$

Equations of motion – Pressure relations

To obtain expressions for the exit pressures, we integrate the Lorenz reciprocal relation over the main channel:

$$\int \nabla \cdot \left(\mathbf{u}^{P_1} \cdot \boldsymbol{\sigma}^{D_1} - \mathbf{u}^{D_1} \cdot \boldsymbol{\sigma}^{P_1} \right) \, \mathrm{d}S = \mathbf{0}$$

Equations of motion – Pressure relations

To obtain expressions for the exit pressures, we integrate the Lorenz reciprocal relation over the main channel:

$$\int_{E_1,E_2,E_3,P,A} \left(\mathbf{u}^{P_1} \cdot \mathbf{f}^{D_1} - \mathbf{u}^{D_1} \cdot \mathbf{f}^{P_1} \right) \, \mathrm{d}s = \mathbf{0}$$

A similar equation holds in the branch

Equations of motion – Pressure relations

To obtain expressions for the exit pressures, we integrate the Lorenz reciprocal relation around the channel:

$$\pi_2 = -Q\Delta p + \frac{1}{Q_1}\left[\int_A \left(\mathbf{f}^{P_1}\cdot\mathbf{u}^{D_1} - \mathbf{u}^{P_1}\cdot\mathbf{f}^{D_1}\right)\mathrm{d}s - \int_P \mathbf{u}^{P_1}\cdot\boldsymbol{\Delta}\mathbf{f}\,\mathrm{d}s + (1-\lambda)\int_P \mathbf{u}^{(ext)}\cdot\mathbf{f}^{P_1}\,\mathrm{d}s\right].$$

$$\pi_3 = \frac{1}{Q_3}\left[\int_A \left(\mathbf{u}^{P_3}\cdot\mathbf{f}^{D_1} - \mathbf{f}^{P_3}\cdot\mathbf{u}^{D_1}\right)\mathrm{d}s + \int_A (\mathbf{f}^{P_1}\cdot\mathbf{u}^{P_3} - \mathbf{u}^{P_1}\cdot\mathbf{f}^{P_3})\,\mathrm{d}s\right]$$

Constitutive equations for a thin elastic shell

Force/moment balance

$$\frac{d\tau}{dl} + \kappa q = -\mathbf{\Delta f} \cdot \mathbf{t}$$

$$\frac{dq}{dl} - \kappa \tau = -\mathbf{\Delta f} \cdot \mathbf{n}$$

$$q = \frac{dm}{dl}$$

Constitutive equations

$$m = E_B \left(\kappa - \kappa_R\right)$$

$$\tau = k \left(\frac{dl}{dl_R} - 1\right)$$

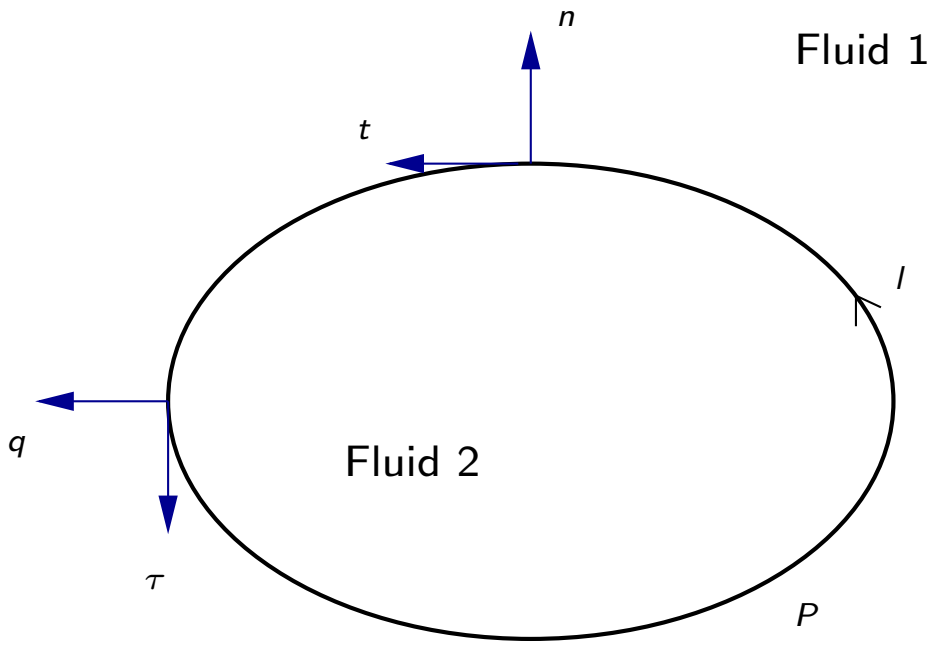


Figure: Elastic cell

Results

Dimensionless parameters

$$\lambda, \quad M = \frac{E_B}{\mu\, Q_1\, d}, \quad W = \frac{k\, d}{\mu\, Q_1},$$

Straight channel

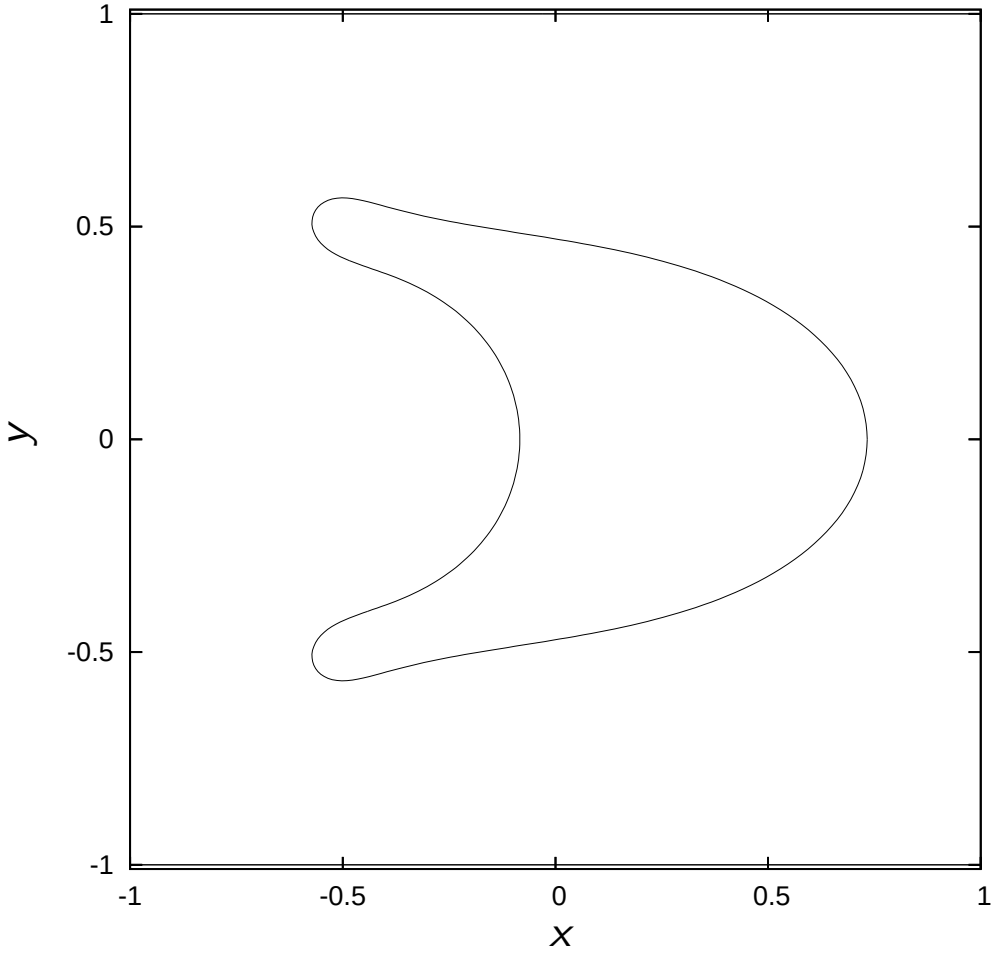


Figure: $\lambda = 1$, $W = 1$ and $a = 0.5\,d$ with $M = 10^{-3}$.

Cell released off-centre: Centroid trajectories

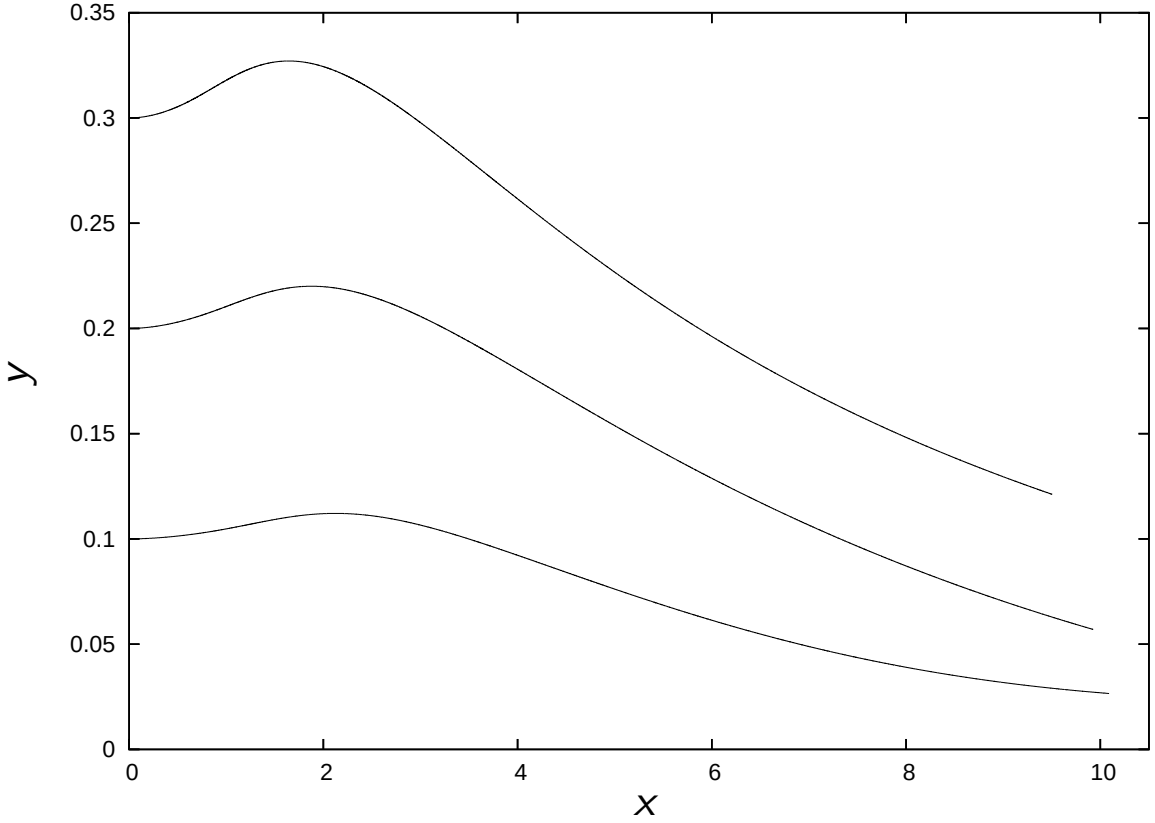
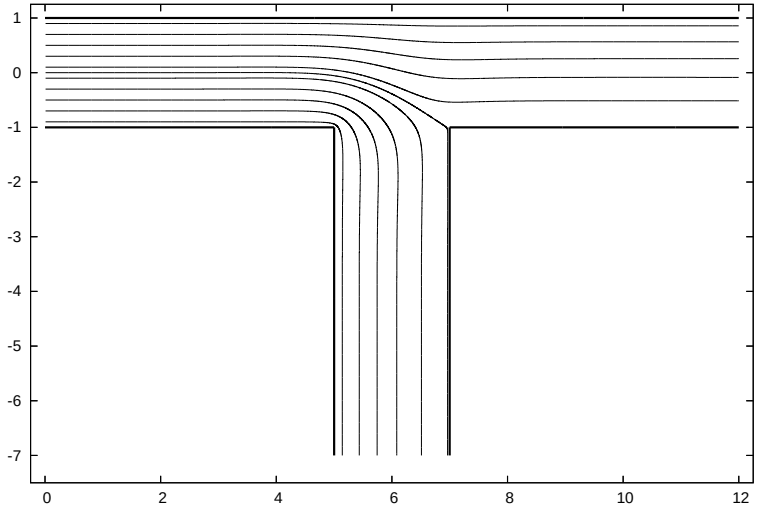
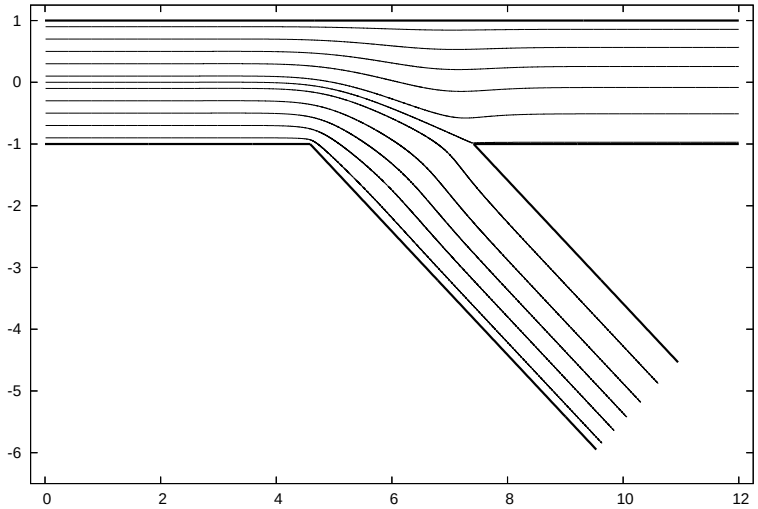
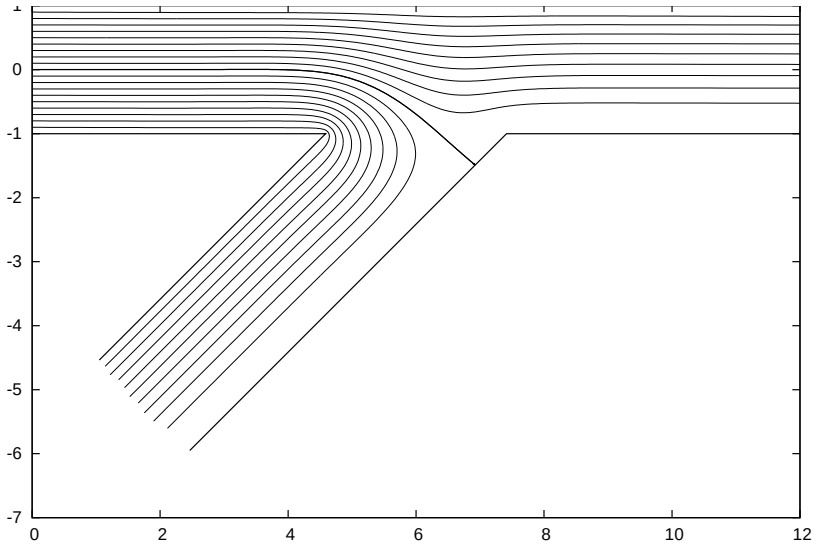


Figure: $a = 0.5d$, $\lambda = 1$, $W = 1$ and $M = 10^{-3}$. Centroid trajectories for circular capsules initially positioned at $y = 0.1\,d$, $0.2\,d$ and $0.3\,d$.

Branching channel: No cell, equal fluxes, $Q_2 = Q_3 = 0.5$







Weak flow in side branch

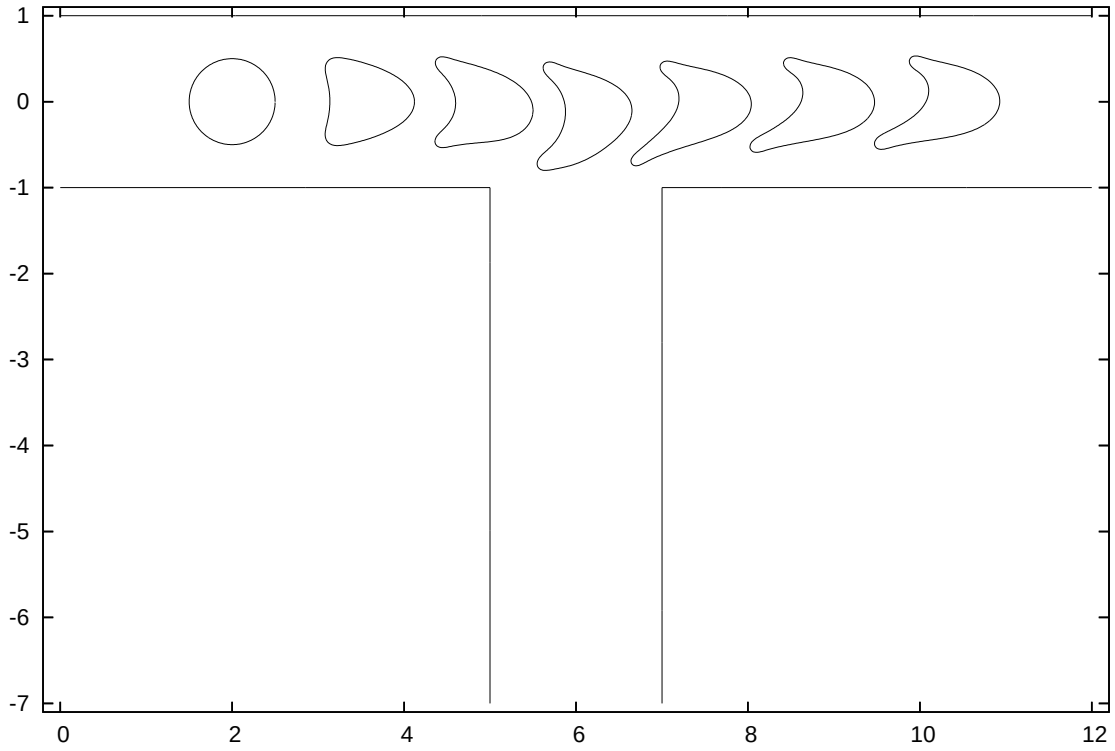


Figure: Capsule journeys for $\lambda = 1$, $a = 0.5\,d$ and $W = 1$ and $Q = 0.9$. At $t = 0$ the capsule centroid is at $(x, y) = (2, 0)$. $M = 10^{-3}$

Cell entering side branch

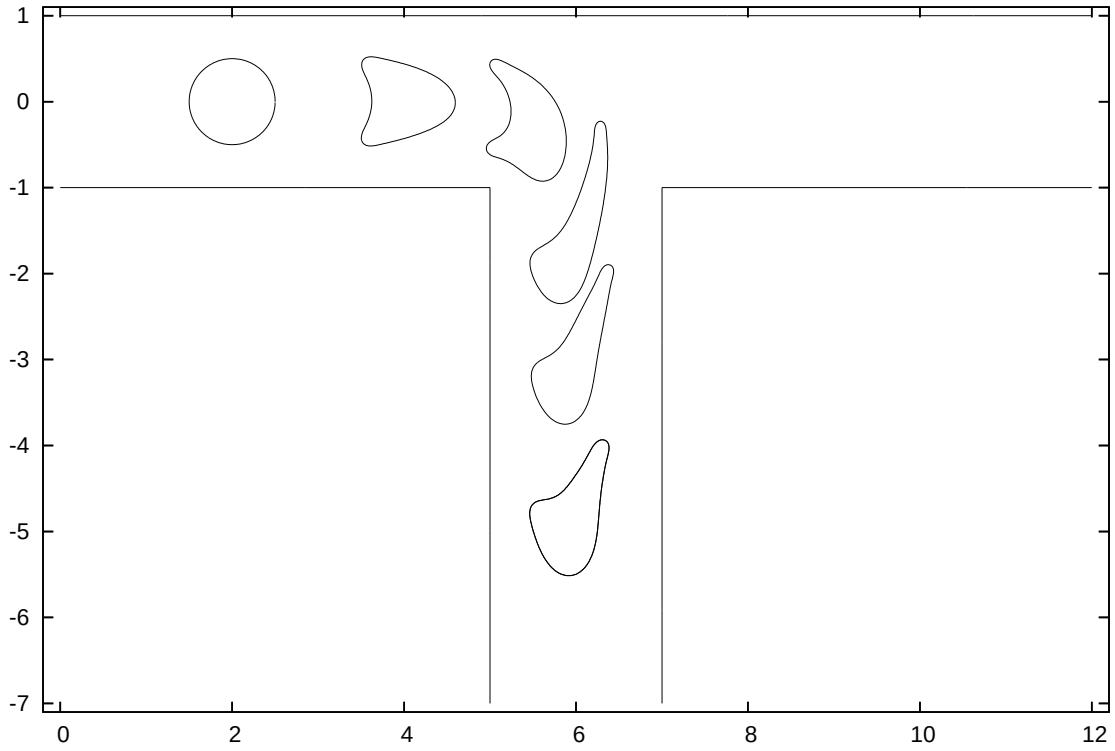


Figure: Capsule journeys for $\lambda = 1$, $a = 0.5\,d$, $W = 1$ and $Q = 0.1$. At $t = 0$ the capsule centroid is at $(x, y) = (2, 0)$. $M = 10^{-3}$.

Narrow side branch

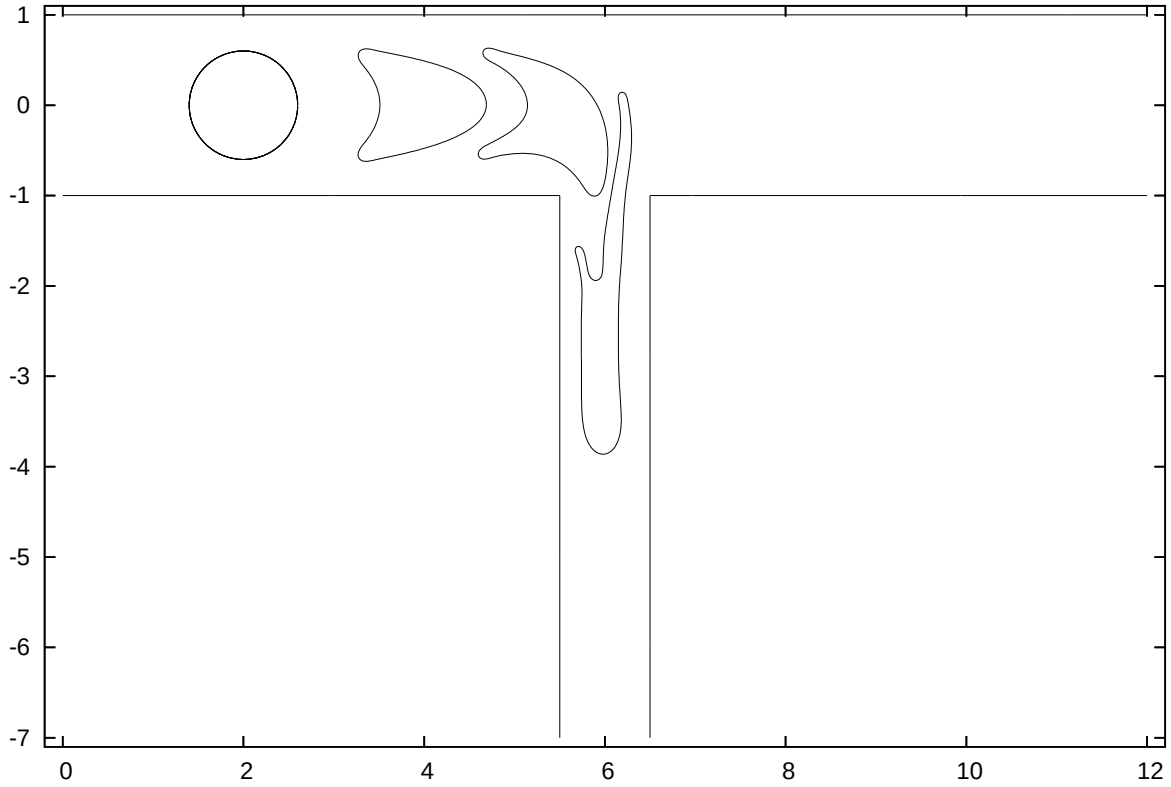


Figure: $\lambda = 1$, $a = 1.1\,d$, $W = 5$, $M = 10^{-3}$, and $Q = 0.1$.

Acute-angled branch: $\alpha = \pi/4$.

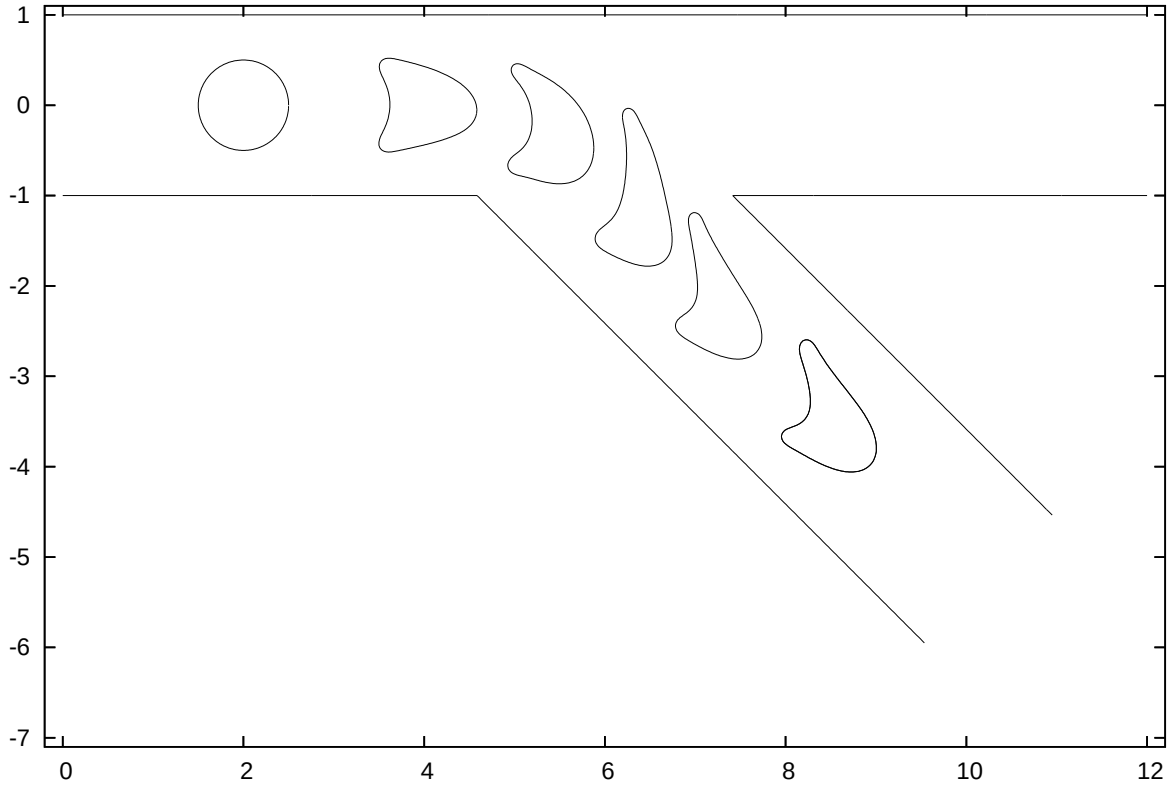


Figure: $\lambda = 1$, $a = 1.1\,d$, $W = 5$, $M = 10^{-3}$, and $Q = 0.1$.

Obtuse-angled narrow side branch

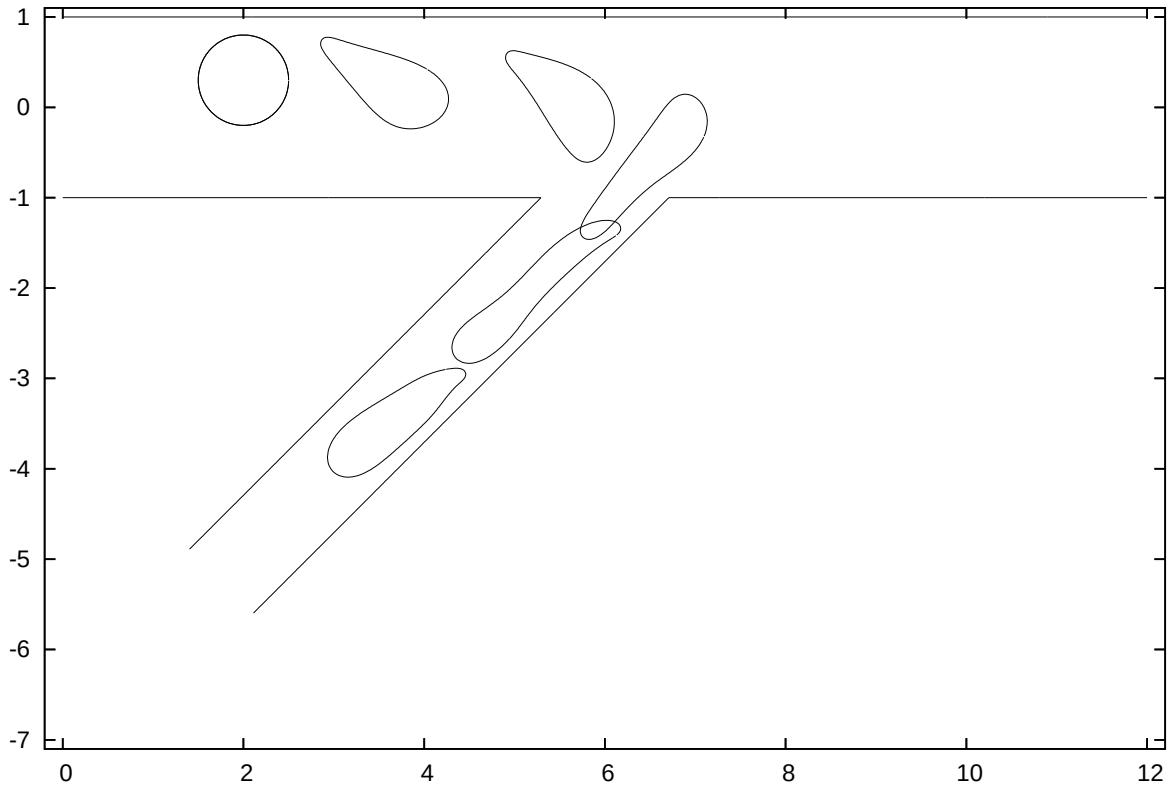


Figure: $\lambda = 1$, $a = 0.5\,d$, $W = 5$, $M = 10^{-3}$, $Q = 0.5$, and $D = 0.5d$, $\alpha = 3\pi/4$. At $t = 0$ circular unstressed shape has centre at $(x_c, y_c) = (2, 0.3)d$.

Strong flow in side branch

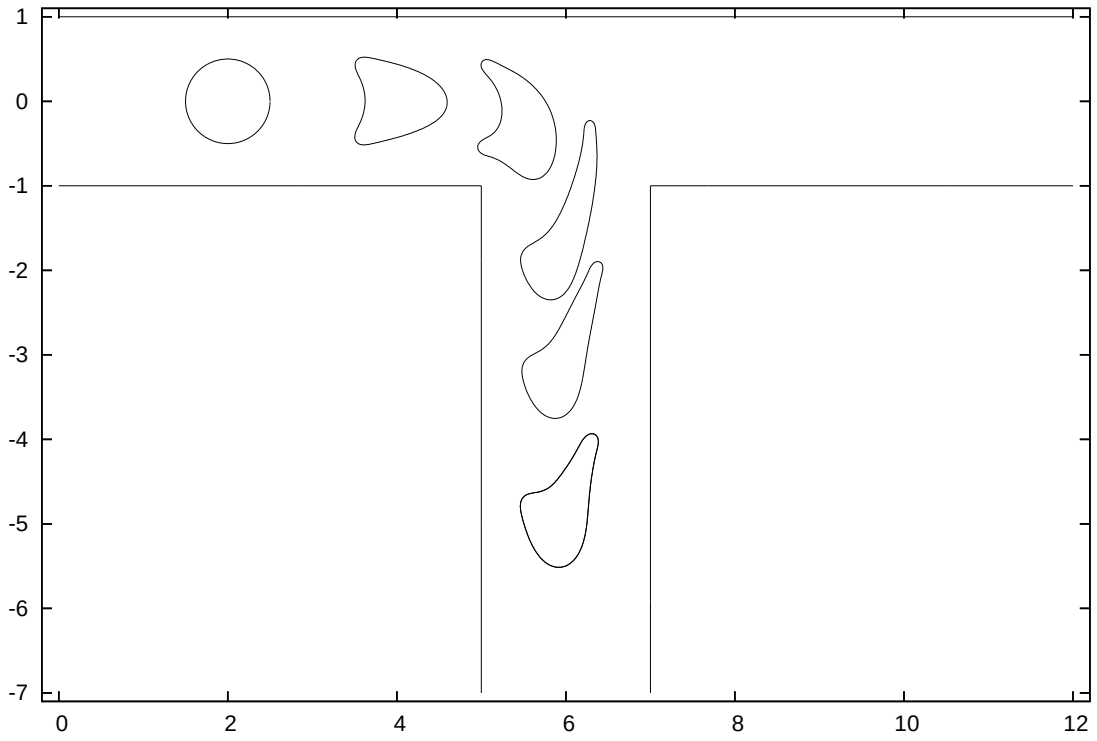


Figure: Capsule journeys for $\lambda = 1$, $a = 0.5\,d$, $W = 1$ and $Q = 0.1$. At $t = 0$ the capsule centroid is at $(x, y) = (2, 0)$. $M = 10^{-3}$.

Typical membrane tensions

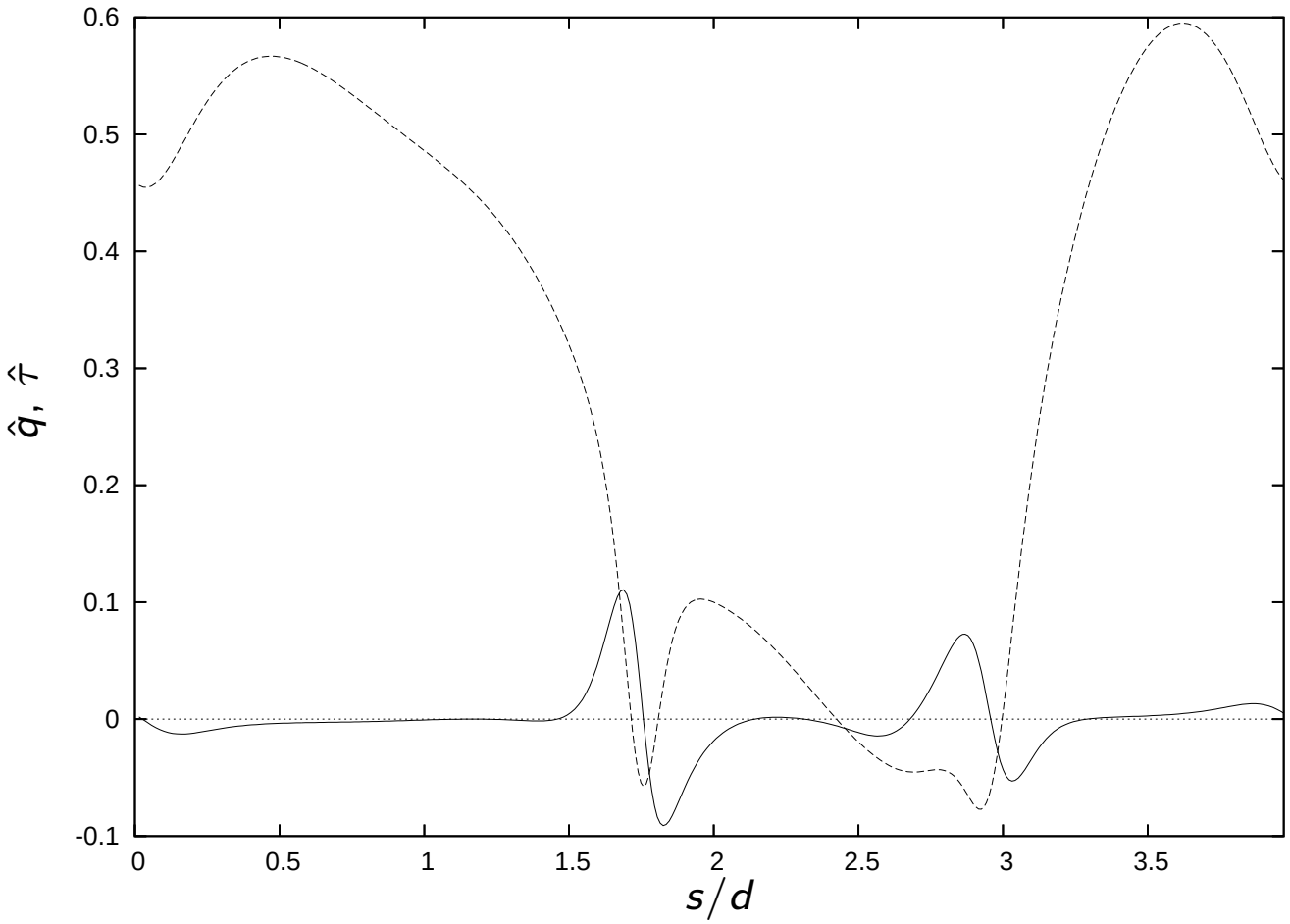
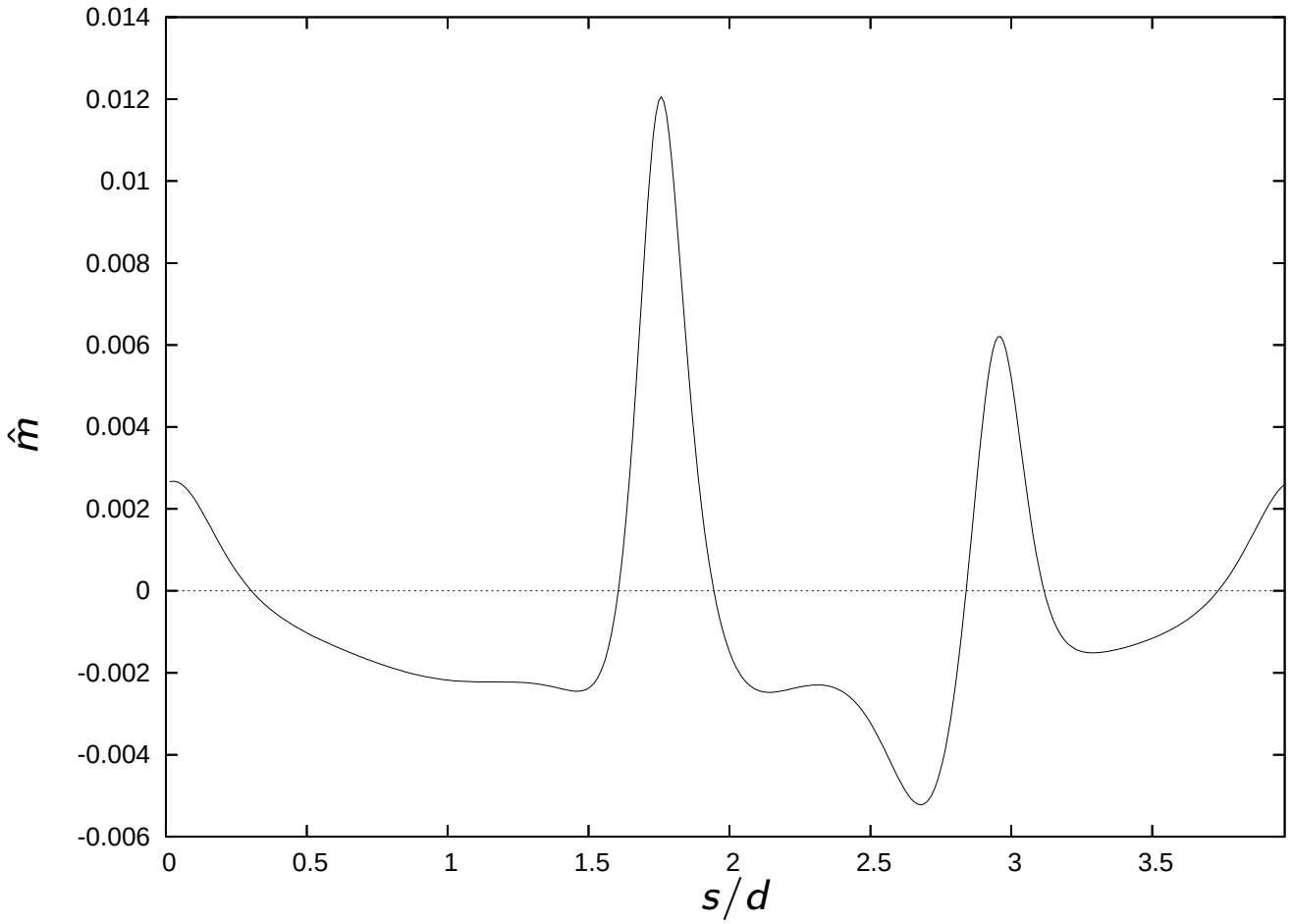
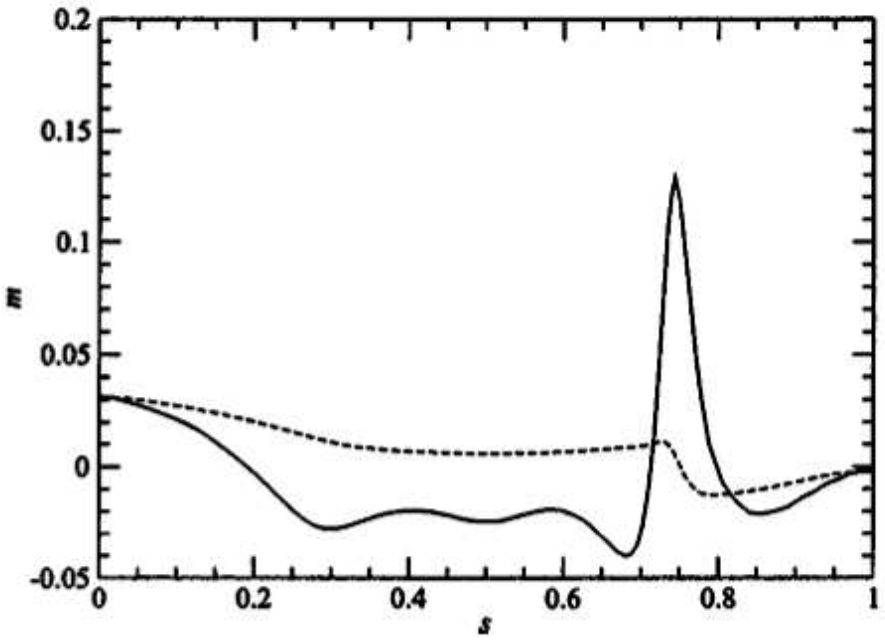


Figure: For last cell shown in previous figure: $\hat{q}$ (solid line), $\hat{\tau}$ (broken line)

Typical membrane bending moment



Typical membrane bending moment



Axisymmetric red blood cell in tube
Pozrikidis (2005), *Phys. Fluids*, **17**(3).

Exit pressures

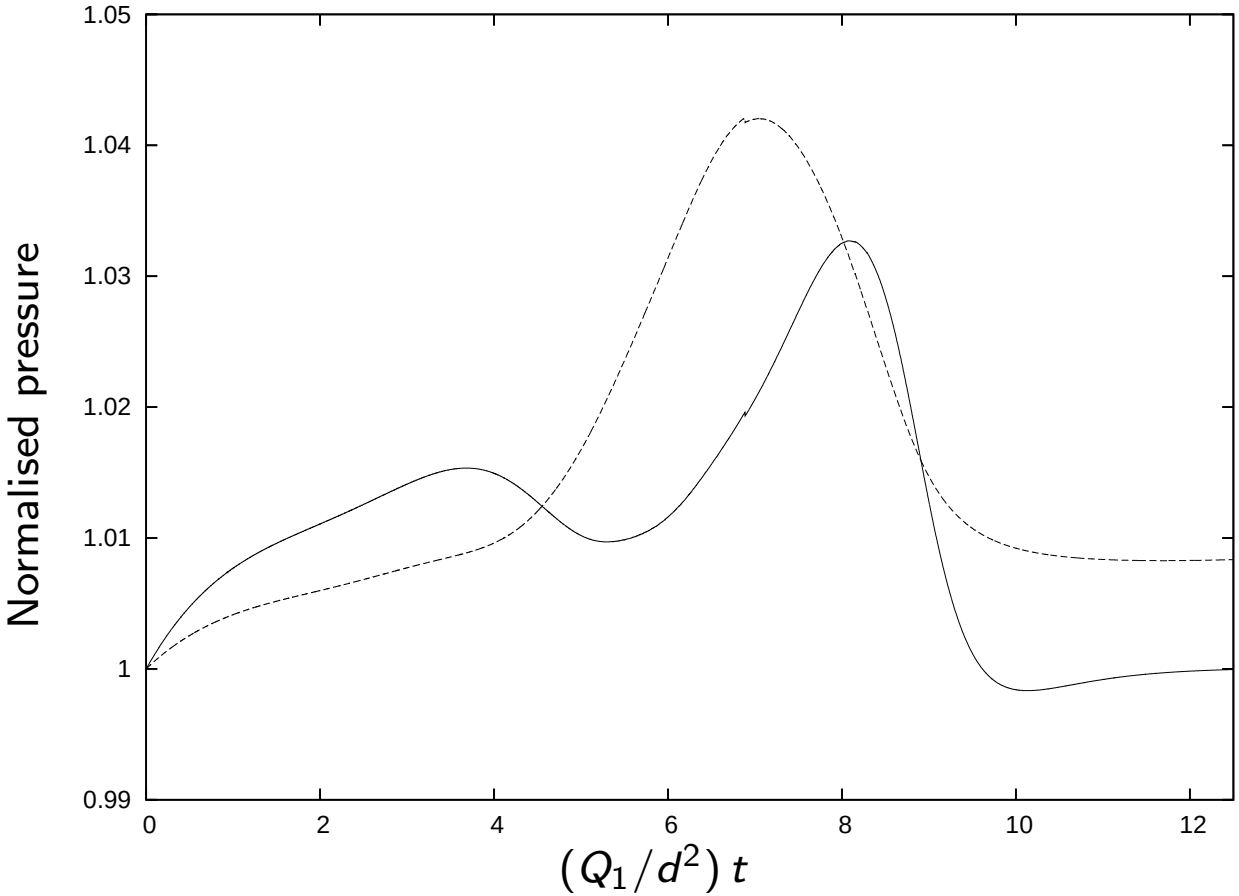
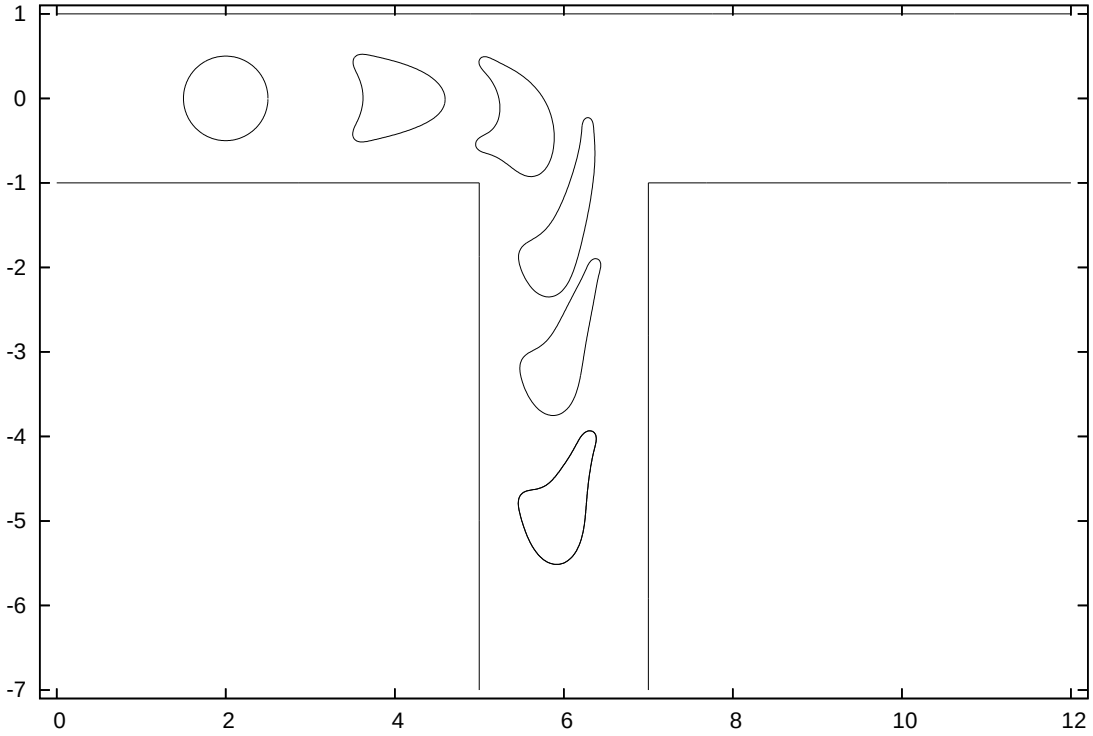


Figure: Normalised pressures $\hat{\pi}_2$ (solid line) and $\hat{\pi}_3$ (dashed line) against time for $Q = 0.1$, $\lambda = 1$, $a = 0.5\,d$, $W = 1$, $M = 10^{-3}$.

Recovery distance



Recovery distance

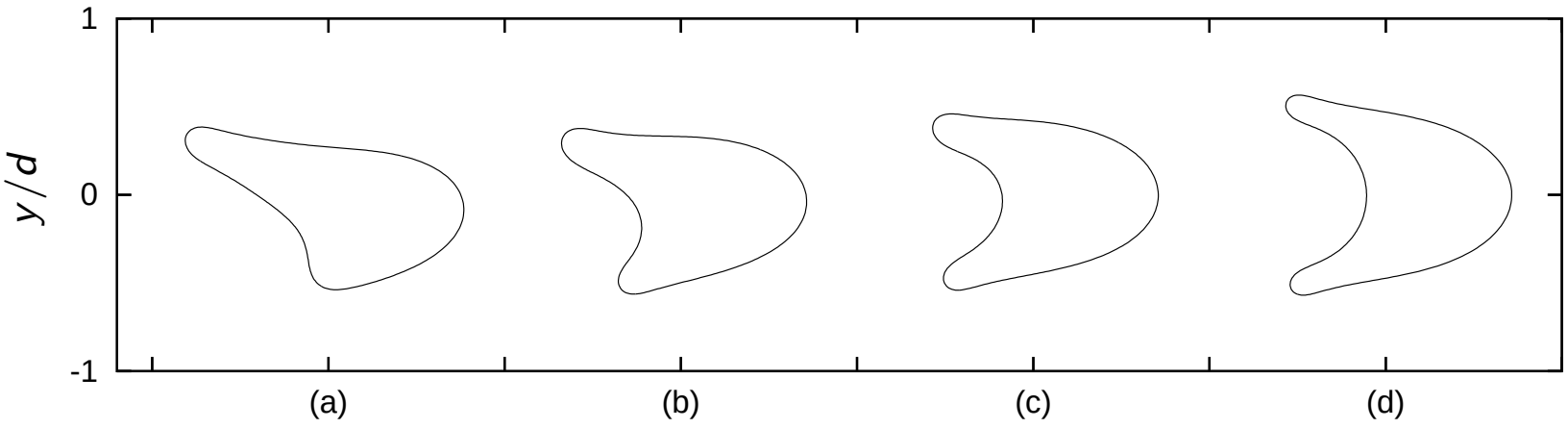
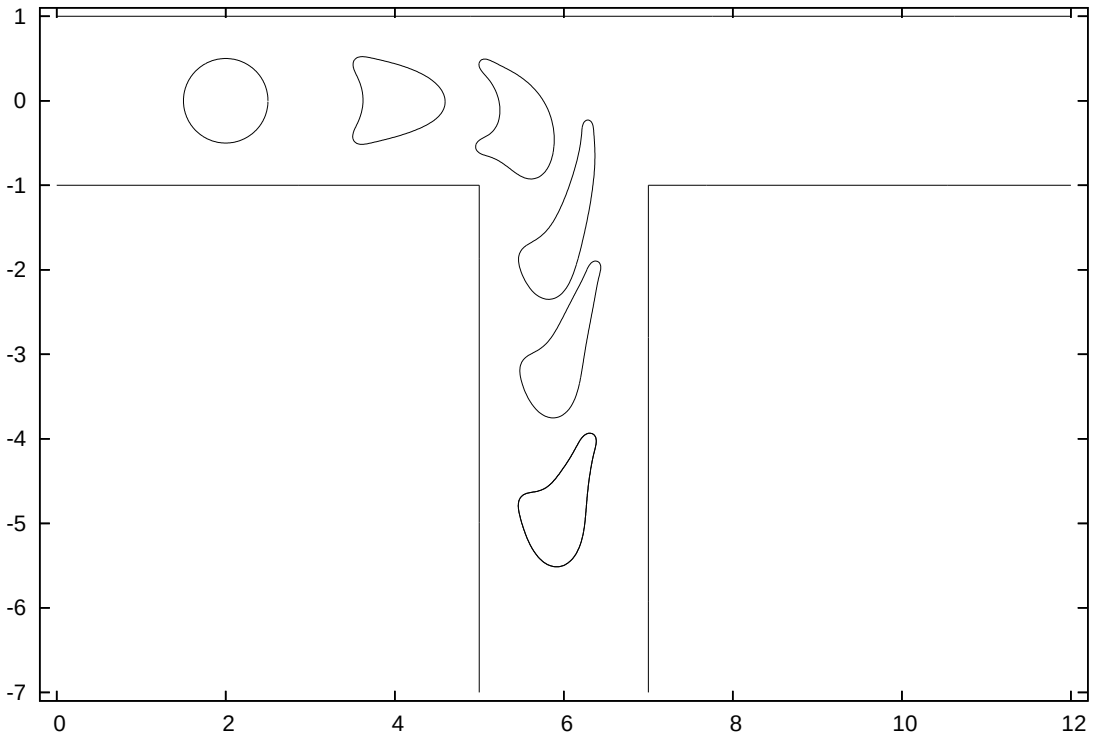


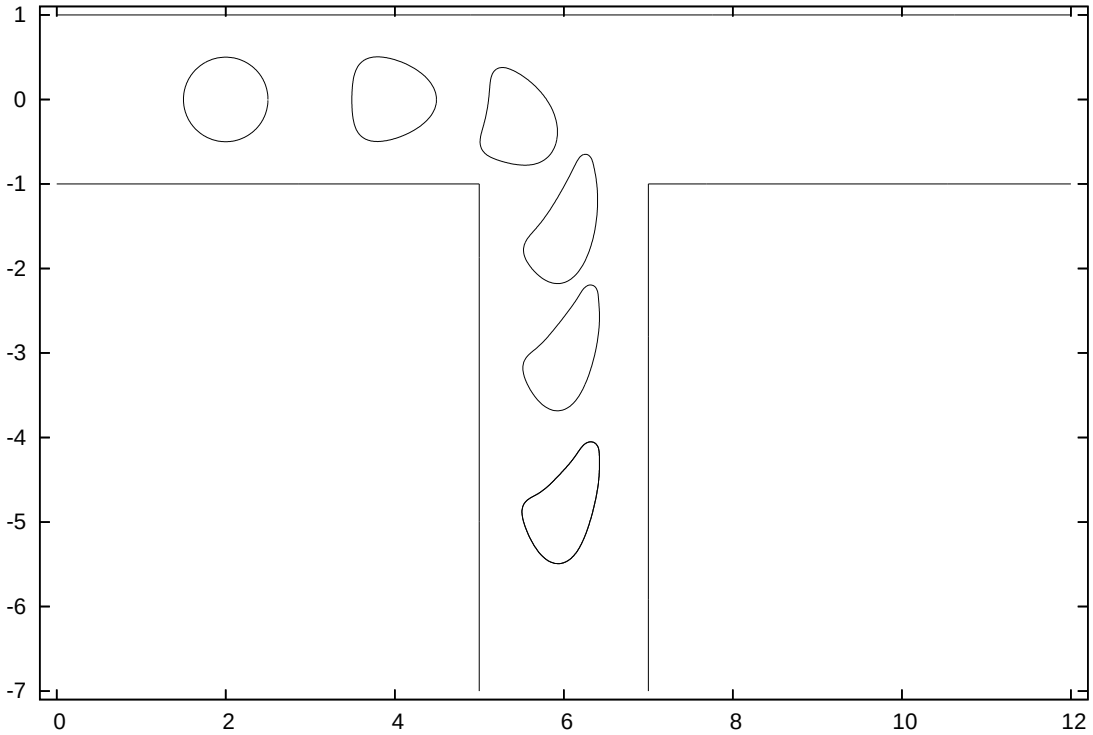
Figure: Evolution of the last capsule shape in previous figure Distances travelled are (a) 0, (b) 5.1a, (c) 12.2a, and (d) 48.9a.

Effect of viscosity ratio



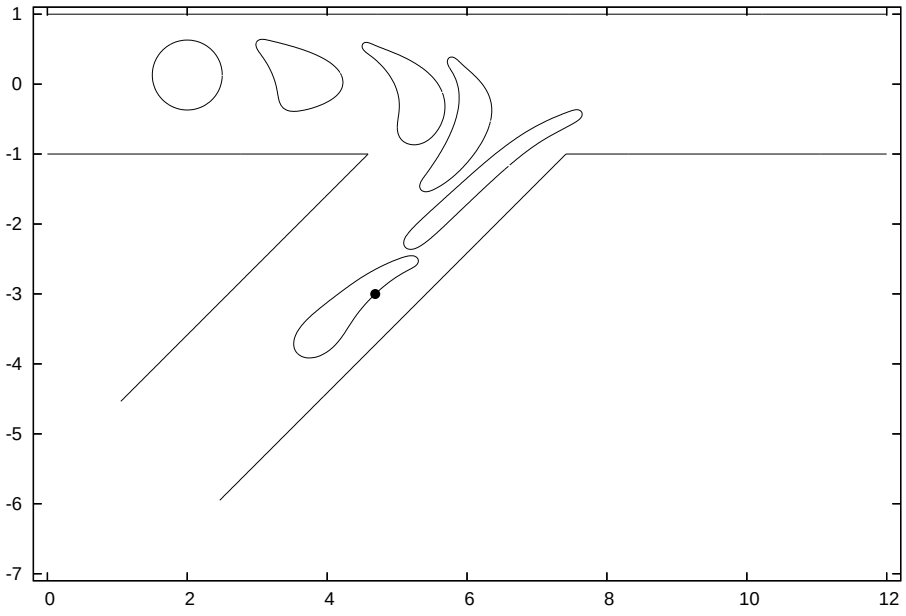
$\lambda = 1,\ Q = 0.1$

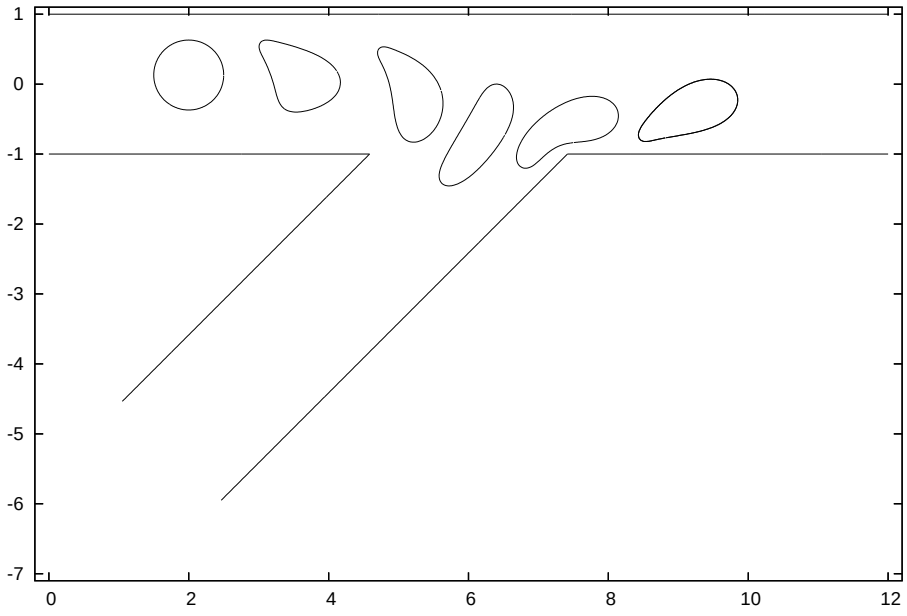
Effect of viscosity ratio



$\lambda = 5$, $Q = 0.1$. Similar effect by increasing W

Path selection

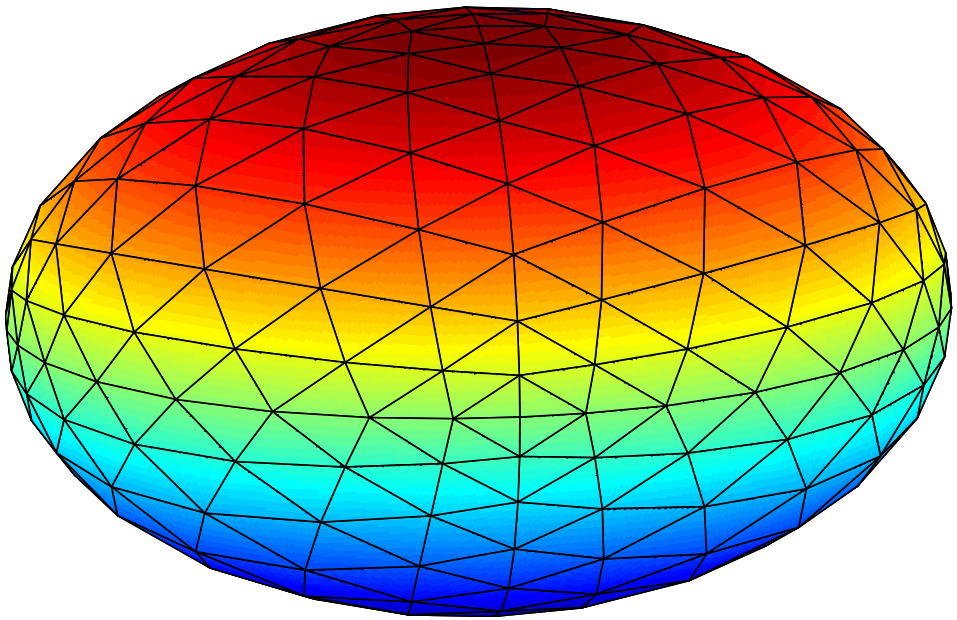




Capsule journeys when $Q = 0.5$, $\lambda = 1$, $M = 10^{-3}$. (left) $W = 1$. (right) $W = 5$.

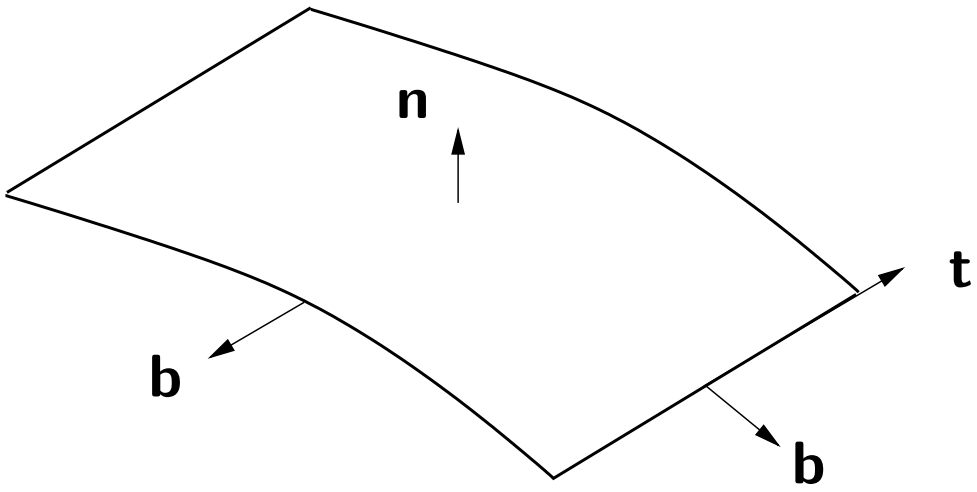
Outline

Capsule mechanics



Capsule mechanics: Equilibrium balance

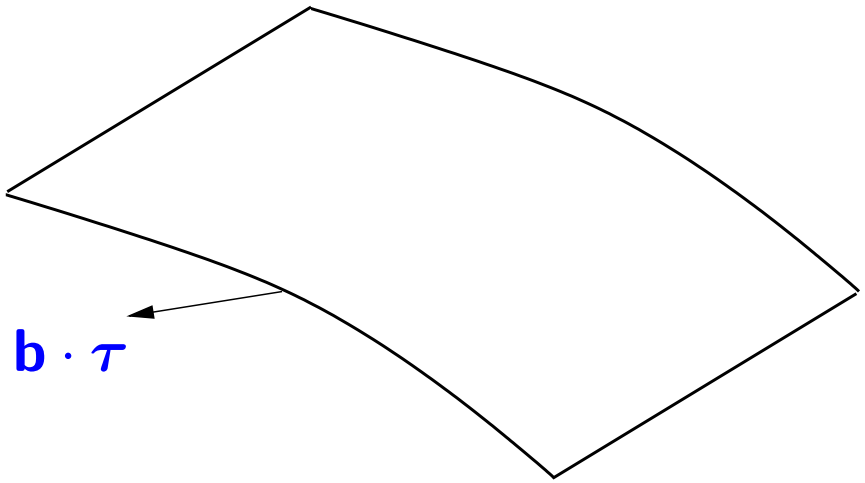
Global Cartesian coordinates : Barthes-Biesel, D. & Rallison (1981) *J. Fluid Mech.*, 113, 251-267.



No bending moments

Capsule mechanics: Equilibrium balance

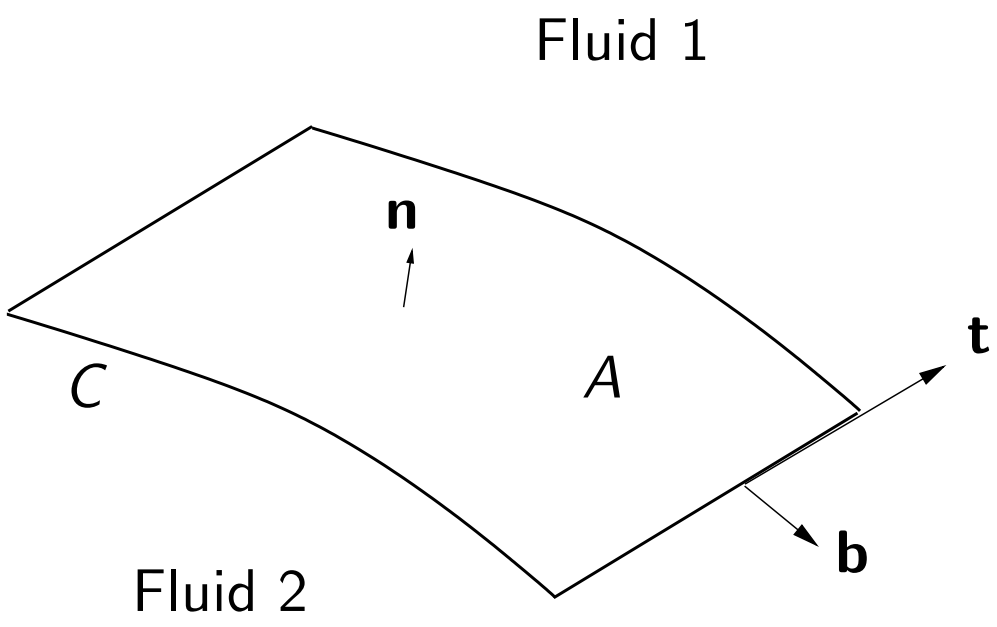
Force on element edge



In-plane tension tensor $\boldsymbol{\tau}$

Capsule mechanics: Equilibrium balance

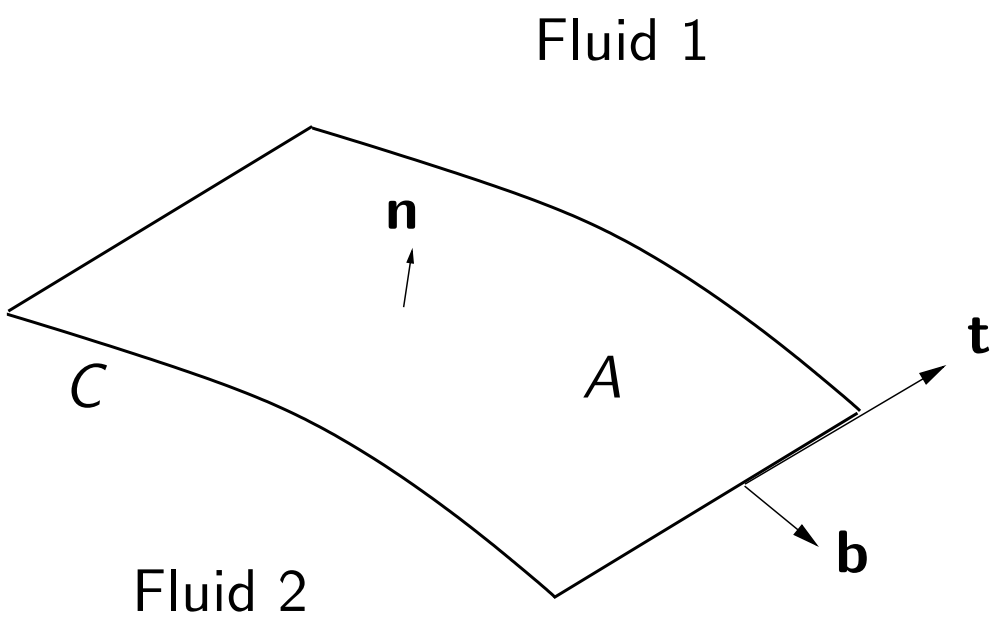
Force balance on membrane patch:



$$\int_A \left[\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right] \cdot \mathbf{n}\, \mathrm{d}S + \int_C \mathbf{b} \cdot \boldsymbol{\tau}\, \mathrm{d}l = \mathbf{0}$$

Capsule mechanics: Equilibrium balance

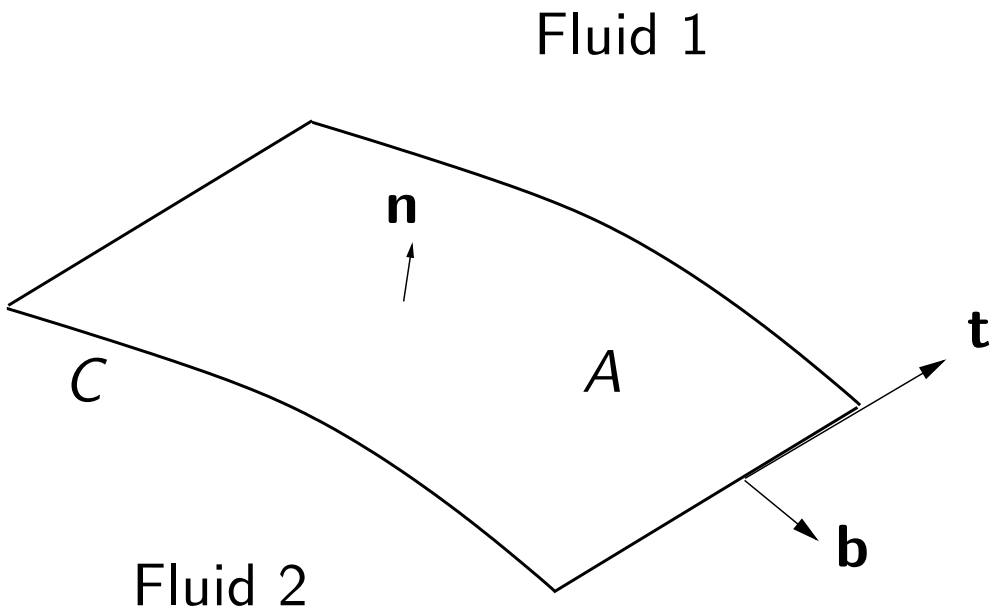
Force balance on membrane patch:



$$\int_A \left[\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right] \cdot \mathbf{n}\, \mathrm{d}S + \int_A \nabla \cdot \boldsymbol{\tau}\, \mathrm{d}S = \mathbf{0}$$

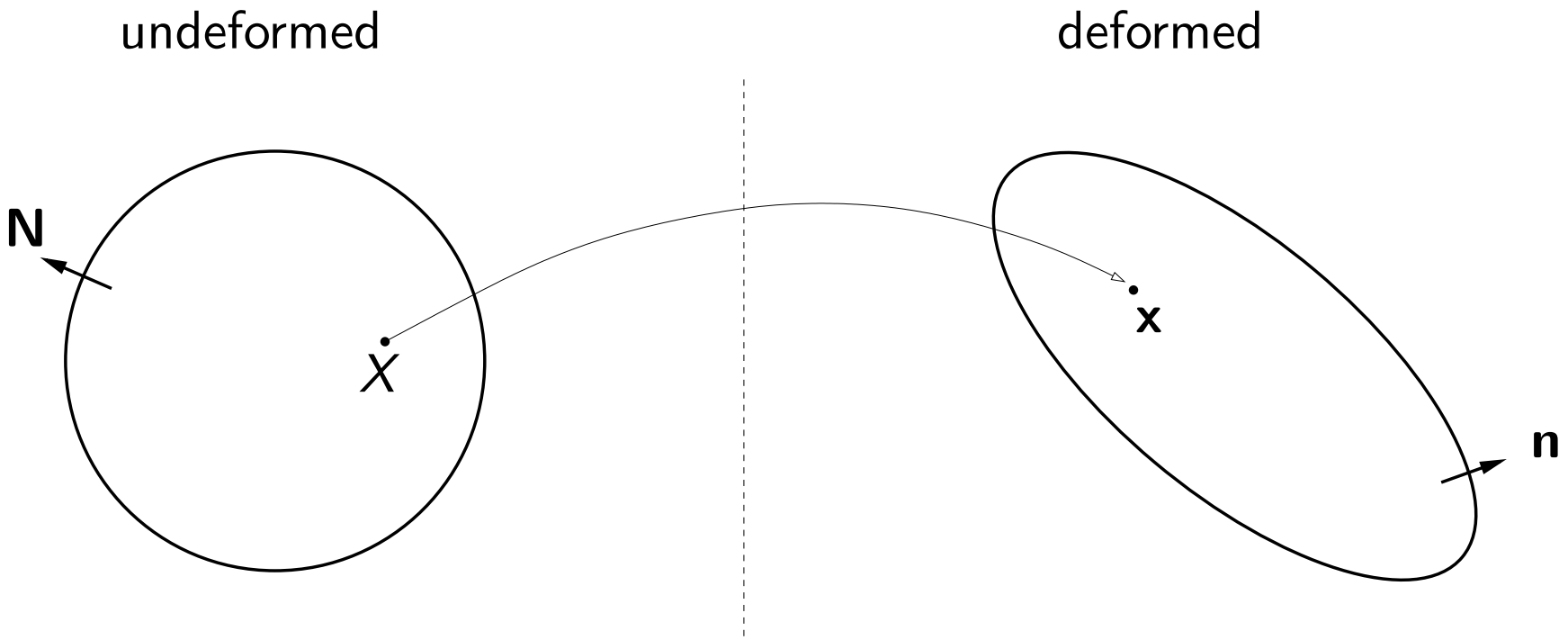
Capsule mechanics: Equilibrium balance

Force balance on membrane patch:



$$\Delta \mathbf{f} + \nabla \cdot \boldsymbol{\tau} = \mathbf{0}, \qquad \Delta \mathbf{f} = \left[\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} \right] \cdot \mathbf{n}$$

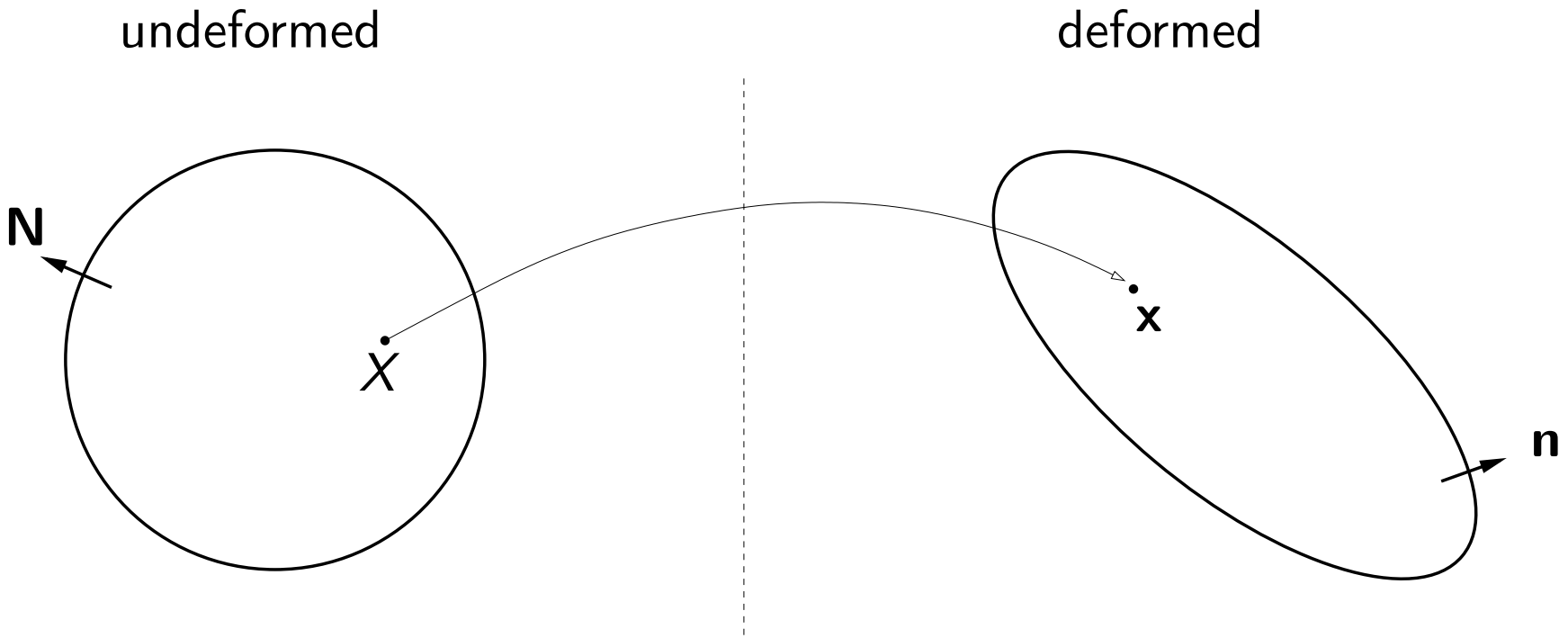
Capsule mechanics: deformation



General form of deformation gradient:

$$\mathbf{F}(t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

Capsule mechanics: deformation



Membrane deformation gradient (Barthes-Biesel & Rallison 1981):

$$\mathbf{A}(t) = (\mathbf{I} - \mathbf{nn}) \cdot \mathbf{F} \cdot (\mathbf{I} - \mathbf{NN})$$

Capsule mechanics: deformation

Membrane deformation gradient:

$$\mathbf{A} = (\mathbf{I} - \mathbf{nn}) \cdot \mathbf{F} \cdot (\mathbf{I} - \mathbf{NN})$$

Projection of surface tangent vector $d\mathbf{X}$

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = \mathbf{A} \cdot d\mathbf{X}$$

Projection of surface normal vector vanishes

$$\mathbf{A} \cdot \mathbf{N} = \mathbf{0}$$

Capsule mechanics: deformation

Membrane deformation gradient:

$$\mathbf{A} = (\mathbf{I} - \mathbf{nn}) \cdot \mathbf{F} \cdot (\mathbf{I} - \mathbf{NN})$$

Projection of surface tangent vector $d\mathbf{X}$

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = \mathbf{A} \cdot d\mathbf{X}$$

Projection of surface normal vector vanishes

$$\mathbf{A} \cdot \mathbf{N} = \mathbf{0}$$

The idea is that membrane fibres pointing in the normal direction do stretch, but do not contribute directly to the elastic tensions.

Capsule mechanics: deformation

To define stretching, we introduce the left Cauchy-Green tensor

$$\mathbf{B} = \mathbf{A} \cdot \mathbf{A}^T \equiv \mathbf{V}^2$$

$\mathbf{B}$ has eigenvalues

$$0, \qquad \lambda_1^2, \qquad \lambda_2^2$$

and eigenvectors

$$\mathbf{n}, \qquad \mathbf{v_1} \qquad \mathbf{v_2}$$

The latter two are the principal directions of stretch

Capsule mechanics: Constitutive equations

Barthes-Biesel & Rallison (1981) showed that

$$\boldsymbol{\tau} = \mathrm{e}^{-\Lambda_1} \left[\frac{\partial W}{\partial \Lambda_1} (\mathbf{I} - \mathbf{nn}) + \frac{\partial W}{\partial \Lambda_2} \mathbf{B} \right]$$

where the invariants

$$\begin{aligned} \Lambda_1 &= \log \lambda_1 \lambda_2, \\ \Lambda_2 &= \frac{1}{2}(\lambda_1^2 + \lambda_2^2) - 1. \end{aligned}$$

W: strain energy function

Capsule mechanics: Constitutive equations

Barthes-Biesel & Rallison (1981) showed that

$$\boldsymbol{\tau} = \mathrm{e}^{-\Lambda_1} \left[\frac{\partial W}{\partial \Lambda_1} (\mathbf{I} - \mathbf{nn}) + \frac{\partial W}{\partial \Lambda_2} \mathbf{B} \right]$$

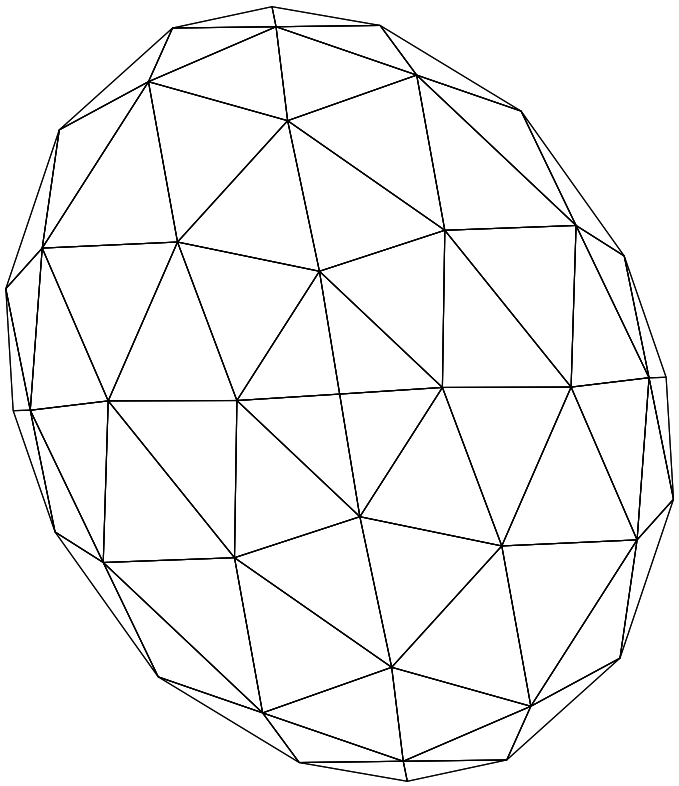
For a red blood cell, Skalak *et al.* (1973), *Biophys J.*, **245**, 245-264 proposed

$$W = \frac{B}{4} \left[2\Lambda_2 (1 + \Lambda_2) + 1 - \mathrm{e}^{2\Lambda_2} \right] + \frac{C}{8} \left[\mathrm{e}^{2\Lambda_1} - 1 \right],$$

where B, C are constants and $B \ll C$.

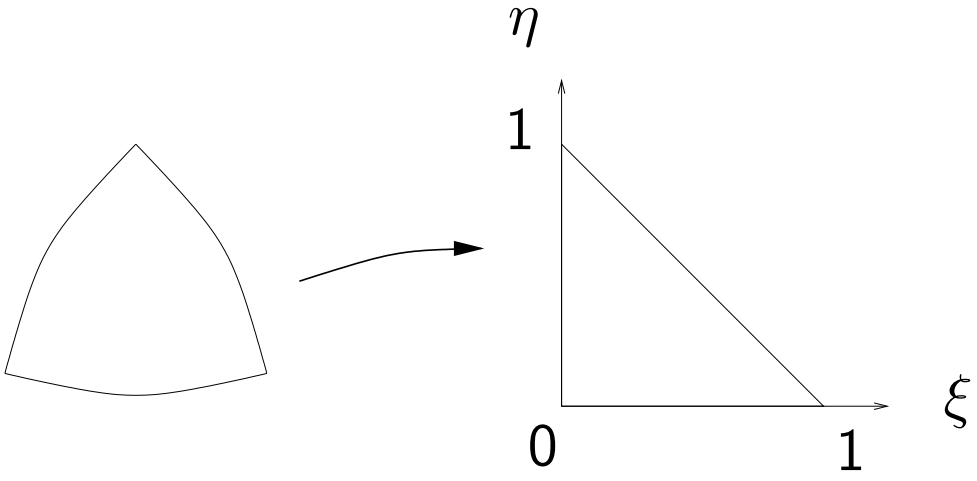
Large C ensures incompressibility of membrane.

Capsule mechanics: Computation



Capsule mechanics: Computation

Ramanujan & Pozrikidis (1998)



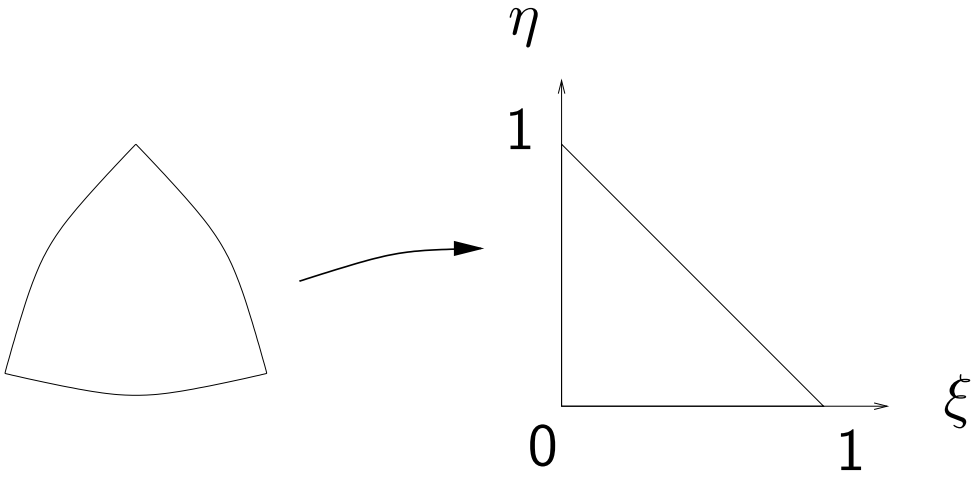
$$\frac{\partial \mathbf{x}}{\partial \xi} = \mathbf{A} \cdot \frac{\partial \mathbf{X}}{\partial \xi}$$

$$\frac{\partial \mathbf{x}}{\partial \eta} = \mathbf{A} \cdot \frac{\partial \mathbf{X}}{\partial \eta}$$

$$\mathbf{0} = \mathbf{A} \cdot \mathbf{N}$$

Capsule mechanics: Computation

Ramanujan & Pozrikidis (1998)



$$\frac{\partial \mathbf{x}}{\partial \xi} = \mathbf{A} \cdot \frac{\partial \mathbf{X}}{\partial \xi}$$

$$\frac{\partial \mathbf{x}}{\partial \eta} = \mathbf{A} \cdot \frac{\partial \mathbf{X}}{\partial \eta}$$

$$\mathbf{0} = \mathbf{A} \cdot \mathbf{N}$$

Solve for **A**

Capsule mechanics: Computation

Once **A** is known:

Capsule mechanics: Computation

Once $\mathbf{A}$ is known:

Compute eigenvalues λ_1, λ_2 of $\mathbf{B} = \mathbf{A} \cdot \mathbf{A}^T$

Capsule mechanics: Computation

Once $\mathbf{A}$ is known:

Compute eigenvalues λ_1, λ_2 of $\mathbf{B} = \mathbf{A} \cdot \mathbf{A}^T$

Compute invariants $\Lambda_1 = \log \lambda_1 \lambda_2$, $\Lambda_2 = \frac{1}{2}(\lambda_1^2 + \lambda_2^2) - 1$.

Capsule mechanics: Computation

Once $\mathbf{A}$ is known:

Compute eigenvalues λ_1, λ_2 of $\mathbf{B} = \mathbf{A} \cdot \mathbf{A}^T$

Compute invariants $\Lambda_1 = \log \lambda_1 \lambda_2$, $\Lambda_2 = \frac{1}{2}(\lambda_1^2 + \lambda_2^2) - 1$.

Compute tension tensor

$$\boldsymbol{\tau} = \mathrm{e}^{-\Lambda_1} \left[\frac{\partial W}{\partial \Lambda_1} (\mathbf{I} - \mathbf{nn}) + \frac{\partial W}{\partial \Lambda_2} \mathbf{B} \right]$$

Capsule mechanics: Computation

Once $\mathbf{A}$ is known:

Compute eigenvalues λ_1, λ_2 of $\mathbf{B} = \mathbf{A} \cdot \mathbf{A}^T$

Compute invariants $\Lambda_1 = \log \lambda_1 \lambda_2$, $\Lambda_2 = \frac{1}{2}(\lambda_1^2 + \lambda_2^2) - 1$.

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$$\boldsymbol{\tau} = \mathrm{e}^{-\Lambda_1} \left[\frac{\partial W}{\partial \Lambda_1} (\mathbf{I} - \mathbf{nn}) + \frac{\partial W}{\partial \Lambda_2} \mathbf{B} \right]$$

Compute $\nabla \cdot \boldsymbol{\tau}$

Capsule mechanics: Computation

Once $\mathbf{A}$ is known:

Compute eigenvalues λ_1, λ_2 of $\mathbf{B} = \mathbf{A} \cdot \mathbf{A}^T$

Compute invariants $\Lambda_1 = \log \lambda_1 \lambda_2$, $\Lambda_2 = \frac{1}{2}(\lambda_1^2 + \lambda_2^2) - 1$.

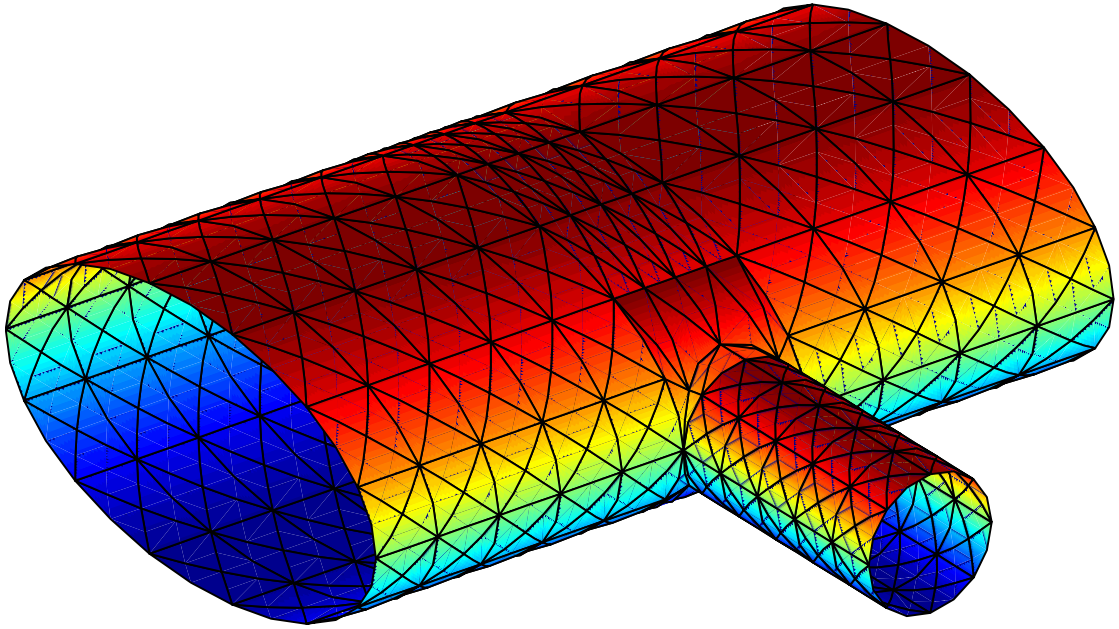
Compute tension tensor

$$\boldsymbol{\tau} = \mathrm{e}^{-\Lambda_1} \left[\frac{\partial W}{\partial \Lambda_1} (\mathbf{I} - \mathbf{nn}) + \frac{\partial W}{\partial \Lambda_2} \mathbf{B} \right]$$

Compute $\nabla \cdot \boldsymbol{\tau}$

Hence $\Delta \mathbf{f} = -\nabla \cdot \boldsymbol{\tau}$

Cell motion through a branching tube



Cell motion through a branching tube

Boundary Integral formulation:

Set viscosity ratio $\lambda = 1$

$\lambda \neq 1$ very computationally expensive

Cell motion through a branching tube

Boundary Integral formulation:

$$u_j(\mathbf{x}_0) = -\frac{1}{8\pi\mu} \int_P \Delta f_i \, G_{ij}(\mathbf{x}, \mathbf{x}_0) \, \mathrm{d}S$$

$$- \frac{1}{8\pi\mu} \int_{E_1, E_2, E_3, C} f_i^{(1)} \, G_{ij}(\mathbf{x}, \mathbf{x}_0) \, \mathrm{d}S$$

$$+ \frac{1}{8\pi} \int_{E_1, E_2, E_3} u_i^{(1)} \, T_{ijk}(\mathbf{x}, \mathbf{x}_0) \, \mathrm{d}S$$

Cell motion through a branching tube

Force balance

Fluid 1

$$\nabla \cdot \boldsymbol{\sigma}^{(1)} = \mathbf{0}$$

Fluid 2

$$\nabla \cdot \boldsymbol{\sigma}^{(2)} = \mathbf{0}$$

Cell motion through a branching tube

Force balance

Fluid 1

$$\int_{E_1,E_2,E_3,C,P} \nabla \cdot \boldsymbol{\sigma}^{(1)} \, \mathrm{d}V = \mathbf{0}$$

Fluid 2

$$\int_{P} \nabla \cdot \boldsymbol{\sigma}^{(2)} \, \mathrm{d}V = \mathbf{0}$$

Cell motion through a branching tube

Force balance

$$p_2 = -\frac{1}{\pi a^2}\left\{ \int_C \mathbf{f}^{(1)} \cdot \mathbf{e}_x \, \mathrm{d}S + \int_P \Delta\mathbf{f} \cdot \mathbf{e}_x \, \mathrm{d}S \right\}$$

$$p_3 = -\frac{1}{\pi b^2}\left\{ \int_C \mathbf{f}^{(1)} \cdot \mathbf{e}_y \, \mathrm{d}S + \int_P \Delta\mathbf{f} \cdot \mathbf{e}_y \, \mathrm{d}S \right\}$$

Outline

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching channel

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching channel

- Boundary integral calculations

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching channel

- Boundary integral calculations
- Domain-decomposition method

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching channel

- Boundary integral calculations
- Domain-decomposition method
- Cell distortion in region of junction

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching channel

- Boundary integral calculations
- Domain-decomposition method
- Cell distortion in region of junction
- Recovery distance

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching channel

- Boundary integral calculations
- Domain-decomposition method
- Cell distortion in region of junction
- Recovery distance
- Path selection may depend on elastic properties

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching tube

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching tube

- Developed a boundary integral formulation

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching tube

- Developed a boundary integral formulation
- Described capsule mechanics

Summary

We have used the boundary integral method to compute cell motion in a branching vessel.

Cell motion in a branching tube

- Developed a boundary integral formulation
- Described capsule mechanics
- Calculations in progress...