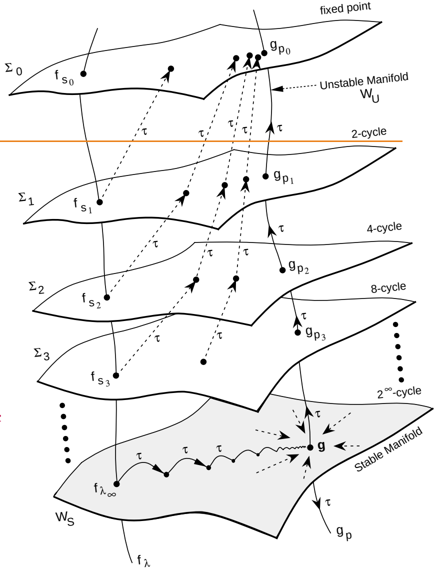
Computer-assisted proofs for renormalisation fixed points



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and Mathematics in Computation'
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Introduction

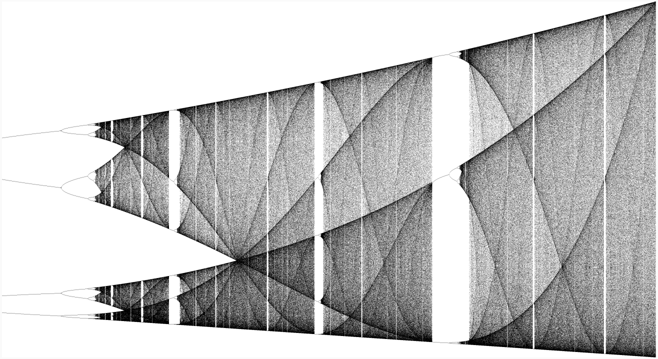
Period-doubling Cascade

One-parameter families of unimodal maps. Prototype example,

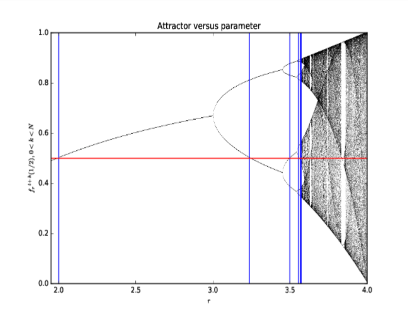
$$f_\mu(x) = 1 - \mu|x|^d.$$

Critical point of degree d (local max) at $x = 0$.

Orbit diagram (e.g., $d = 2$)



Universal Features



Feigenbaum (L), Cvitanović (R)

For degree $d = 2$ critical point: **universal** constants

$$\delta_2 := \lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} = 4.669\,201\,609\,102\,990\,671\ \ldots,$$

$$\alpha_2 = -2.\,50290\,78750\,95892\,82228\,39028\,73218\ \ldots$$

[Feigenbaum 1978; Coullet, Tresser 1978]

Renormalisation operator [Cvitanović-Feigenbaum]

$$Rg(x) = a^{-1}g(g(ax)),$$

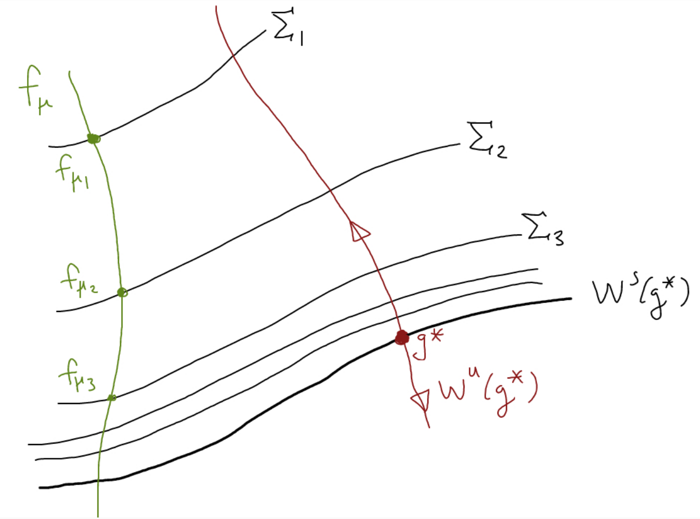
where $a := g(1)$, chosen to preserve the normalisation $g(0) = 1$.

(Main) Renormalisation Conjectures

There exists a nontrivial **hyperbolic renormalisation fixed point** g^* (with $\alpha = 1/g^*(1)$).

The spectrum of the linearisation $DR(g^*)$ of the operator has a single (non-coordinate-change) expanding eigenvalue (δ).

Sketch in a suitable space of functions



Σ_n — surface of functions with a superstable period 2^n orbit.

Some Previous Results

Analytical proofs including many recent generalisations:

1986 Epstein
1987 Sullivan
1996 McMullen
1999 Lyubich
2006 Faria et al.
2011 Avila et al.
2018 Gorbovickis et al.

First approaches via **rigorous computer-assisted proofs**.

1982 Lanford [above, left]

(Other scenarios:)
1984 Eckmann, Koch, Wittwer
1985 Mestel
1999 Stirnemann

The renormalisation fixed point
(one computational approach)

Renormalisation operator

Renormalisation operator, R

$$Rg(x) := a^{-1}g(g(ax)), \quad \text{where } a := g(1).$$

We seek a nontrivial solution, g^*, to the functional equation

$$g^*(x) = a^{-1}g^*(g^*(ax)), \quad \text{with } a = g^*(1).$$

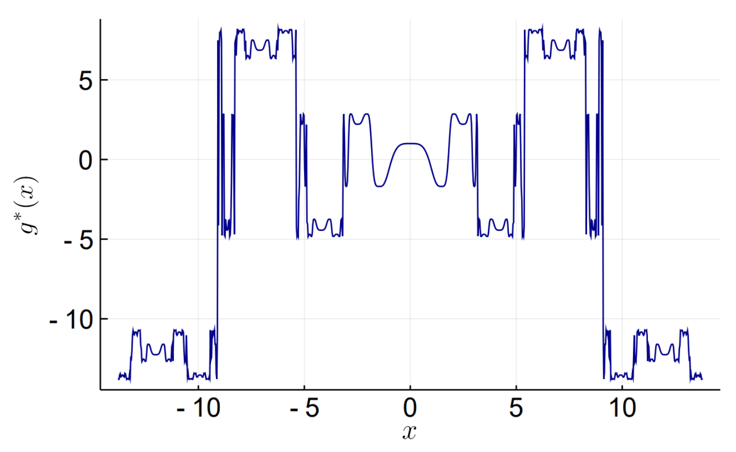
Ansatz $g(x) = G(x^d)$: Let $X = x^d =: Q(x)$ and write

$$g(x) = G(Q(x)) = G(X).$$

Corresponding operator, T:

$$TG(X) := a^{-1}G(Q(G(Q(a)X))), \quad \text{where } a := G(1).$$

Approximate fixed point function $g^0(x) = G^0(x^d)$



Nonrigorous approximation to $g^*(z)$ for $d = 4$.

A suitable function space: the (poly)disc algebra

Consider the space $\mathcal{A} = \mathscr{A}(\Omega)$ of functions analytic on a disc

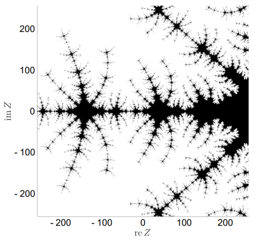
$$\Omega = D(c, r) := \{z \in \mathbb{C} : \ |z - c| < r\}$$

and continuous on its closure, $\overline{\Omega}$, with (finite) ℓ_1-norm:

$$f(z) = \sum_{k=0}^{\infty} a_k \left(\frac{z-c}{r}\right)^k, \quad \|f\| = \|f\|_1 := \sum_{k=0}^{\infty} |a_k|.$$

Operations:

$$\begin{aligned}(f+g)(z) &:= f(z) + g(z),\\ (af)(z) &:= af(z),\\ (f \cdot g)(z) &:= f(z)\,g(z),\\ (f \circ g)(z) &:= f(g(z)).\end{aligned}$$



$\mathcal{A}$ is a Banach space with Schauder basis, e.g., $e_j(z) = \left(\frac{z-c}{r}\right)^j$.

Rigorous computations in the function space?

Truncation

Two fundamental hurdles:
(a) the function space is infinite-dimensional
(b) computer arithmetic is not exact

Work with power series truncated after some fixed degree N.

$$\ell_1 \cong \mathbb{R}^{N+1} \oplus \ell_1.$$

Let $P\mathcal{A}$ and $H\mathcal{A} = (I - P)\mathcal{A}$ denote canonical projections onto the polynomial part and high-order part.

Write $f \in \mathcal{A}$ as

$$f = f_P + f_H,$$

with $f_H \in H\mathcal{A}$ and $f_P \in P\mathcal{A}$ where

$$f_P(x) = \sum_{k=0}^{N} a_k \left(\frac{x-c}{r}\right)^k.$$

Rigorous framework

Idea: use sets of objects, with bounds represented exactly.

Let $v_P = ([b_0, c_0], \ldots, [b_N, c_N])$ be a vector of intervals.
Let $v_H, v_G \geqslant 0$.

Define the (convex, closed) **function ball** $B(v_P, v_H, v_G) \subset \mathcal{A}$ by

$$B(v_P, v_H, v_G) := \Big\{ f \in \mathcal{A} : \\ f = f_P + f_H + f_G, \\ f_P \in P\mathcal{A},\ f_H \in H\mathcal{A},\ f_G \in \mathcal{A}, \\ f_P(z) = \sum_{k=0}^{N} a_k \left(\frac{z-c}{r} \right)^k, a_k \in [b_k, c_k], \\ \|f_H\| \leqslant v_H, \\ \|f_G\| \leqslant v_G \Big\}.$$

Rigorous framework

Use **computer-representable numbers** for bounds (v_P, v_H, v_G).

For each binary operation $\oplus$, design a version, $\oplus_b$, acting on bounds v, w such that

$$\forall f \in B(v),\ g \in B(w): \qquad f \oplus g \in B(v \oplus_b w).$$

Analogy (interval arithmetic): $x \in X = [a, b]$ and $y \in Y = [c, d]$

$$\begin{aligned} x + y \in X + Y &= \{z = u + v : u \in X, v \in Y\} \\ &= [a + c, b + d] \\ &\subseteq [a +_\downarrow c, b +_\uparrow d], \end{aligned}$$

where $+_\downarrow, +_\uparrow$ always yield representable lower/upper bounds.

Operations on $\mathcal{A}$: $f + g,\ af,\ f \cdot g,\ f \circ g,\ \|f\|,\ \not{f'},\ f' \circ g.$

[Moore 1966. Eckmann, Koch, Wittwer 1982. Kaucher et al 2014.]

Bounds on the fixed point

Newton's method as a contraction mapping

Newton operator for fixed points, $T(G) - G = 0$:

$$\Phi : G \mapsto G - [DT(G) - I]^{-1}(T(G) - G),$$

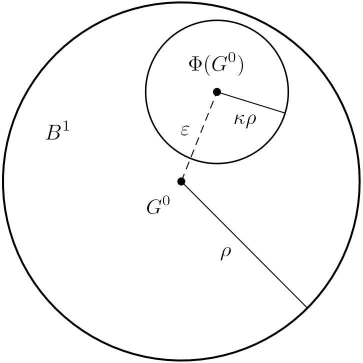
Idea: show that Φ is a contraction map on ball $B^1 = B(G^0; 0, \rho)$, for which we first prove that T is C^∞ and $DT(G)$ is compact.

3 ingredients:

1. $\|\Phi(G^0) - G^0\| \leqslant \varepsilon$,
2. $\exists \kappa < 1 :\ \forall F, G \in B^1$:

   $\|\Phi(F) - \Phi(G)\| \leqslant \kappa \|F - G\|$,
3. $\varepsilon < \rho(1 - \kappa)$.

$(1, 2, 3) \implies \Phi(B^1) \subset B^1$.



Bounds

Bound 2: uniform contractivity

Idea: bound $\|D\Phi(G)\| \leqslant \kappa < 1$, $\forall G \in B^1$,
and use the **mean value theorem**.

Use 'maximum column sum norm' (countable basis, $\{e_j\}_{j \geqslant 0}$):

$$\begin{aligned} \|D\Phi(G)\| &= \sup_{\|\delta G\|=1} \|D\Phi(G)\delta G\| \\ &\leqslant \sup_{j \geqslant 0} \left\|D\Phi(G)e_j\right\| \leqslant \kappa < 1. \end{aligned}$$

Compute $D\Phi(B^1)E_j$ for balls $E_j := B(e_j;\, 0,\, 0)$ for $j = 0, \ldots, N$.

Problem: infinitely-many basis elements remain.

Idea: compute $D\Phi(B^1)E_H$ for the single ball $E_H := B(0;\, 1,\, 0)$.

Domain extension and compactness

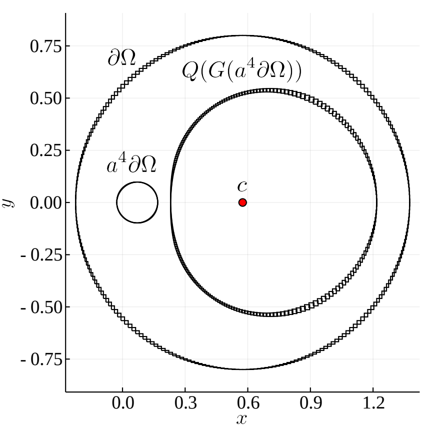
Important: we need that T is **well-defined and differentiable**, with **compact derivative**, $DT(G)$, for all G in B^1.

$$TG(X) := a^{-1}G(Q(G(Q(a)X))), \quad \text{where } a := G(1).$$

Domain extension:
For all $G \in B^1$

$$Q(a)\overline{\Omega} \subset \Omega,$$
$$Q(G(Q(a)\overline{\Omega})) \subset \Omega.$$

For $d = 4$, we may choose $\Omega = D(0.5754, 0.8)$.



Newton-like operator

Consider Newton operator for the fixed-point problem

$$\phi : G \mapsto G - [DT(G) - I]^{-1}(T(G) - G).$$

Simplify: choose an invertible linear operator Λ and consider

$$\Phi : G \mapsto G - \Lambda(T(G) - G),$$

We choose

$$\Lambda \simeq [DT(G) - I]^{-1}.$$

Idea: Approximate $DT(G)$ by a **fixed** linear operator

$$\Delta \simeq DT(G^0) \qquad \text{with} \quad \Delta H \mathcal{A} = 0.$$

The Frechet derivative of the Newton-like operator Φ

$$D\Phi(G) : \delta G \mapsto \delta G - \Lambda[DT(G)\delta G - \delta G].$$

Detail: Dependency problem for fixed point

Newton-like operator

$$\Phi : G \mapsto G - \Lambda(T(G) - G),$$

For the choice $\Lambda = (\Delta - I)^{-1}$, $\Lambda(\delta b_H) = (-I)\delta b_H$ for $\delta b_H \in H\mathcal{A}$.

$$D\Phi(B^1)\delta b_H = \delta b_H - \Lambda\left[DT(B^1)\delta b_H - \delta b_H\right] \quad (1)$$

$$= \delta b_H - \Lambda\left[DT(B^1)\delta b_H\right] - \delta b_H \quad (2)$$

$$= -\Lambda\left[DT(B^1)\delta b_H\right]. \quad (3)$$

Computing $\|D\Phi(B^1)E_H\|$ naively, using (1) instead of (3) gives $\kappa > 2$, even where $D\Phi(B^1)$ is contractive, due to implicit presence of 'uncancelled' terms $\delta b_H - \delta b_H$ in (2). (Operands in expressions of the form $\|f - g\|$ are treated as independent (high-order) functions; analogy: $[0,1] - [0,1] = [-1,1]$ for $x - x$.)

Frechet derivative $DT(G)$

For the operator T:

$$T(G)(X) = a^{-1}G(Q(G(Q(a)x))).$$

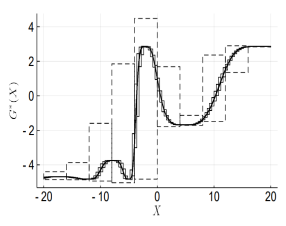
We obtain (expanded to show linearity in δG)

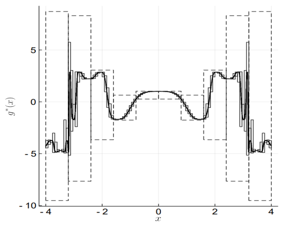
$$
\begin{aligned}
&DT(G) : \delta G \mapsto \\
&- a^{-2}\delta a \cdot G(Q(G(Q(a)X))) \\
&+ a^{-1}\delta G(Q(G(Q(a)X))) \\
&+ a^{-1}G'(Q(G(Q(a)X))) \cdot Q'(G(Q(a)X)) \cdot \delta G(Q(a)X) \\
&+ a^{-1}G'(Q(G(Q(a)X))) \cdot Q'(G(Q(a)X)) \cdot G'(Q(a)X) \cdot Q'(a)\delta a \cdot X.
\end{aligned}
$$

For $a = G(1)$ we have $\delta a = \delta G(1)$.

Existence and local uniqueness

Example ($d = 4$): For a particular disc Ω, and truncation degree $N = 40$ for G (thus 160 for g) and working to 40sf, we obtain $\varepsilon = 1.59 \times 10^{-21}$, and choosing $\rho = 10^{-20}$ gives $\kappa = 6.88 \times 10^{-3}$.

G: 

g: 

Rigorous bounds on universal constants:

$$a_4 = g(1) = -0.5916099166\ 3443815013\ \ldots$$

$$\alpha_4 = 1/g(1) = -1.6903029714\ 0524485334\ \ldots$$

Bounding eigenfunctions and eigenvalues

The spectrum of $DT(G^*)$

Compactness of a bounded linear operator $L \in \mathscr{B}(\mathcal{A}, \mathcal{A})$ implies that the spectrum of L consists of 0 together with **a countable set of isolated eigenvalues of finite multiplicity** (which accumulate at 0) — a generalisation of square matrices.

The spectrum of $DT(G^*)$ has 2 eigenvalues in the complement of the closed unit disc,

$$\alpha^d, \delta.$$

Note:

α^d is a coordinate-change eigenvalue
(where $\alpha = 1/G^*(1)$ is responsible for scaling in x).

δ is responsible for universal scaling in parameter μ.

We can bound the spectrum using contracted matrices.

Detail: Bounding the spectrum (directly)

1. Change coords to make $DT(G)$ close to diagonal ($\forall G \in B^1$)
$$DT(G) \mapsto C^{-1} DT(G) C =: L.$$
2. Choose $m \leqslant N$. Find an $(m+1) \times (m+1)$ 'contracted (rectangle) matrix', $M \ni L$. If $r = [a,b] + i[c,d] \subset \mathbb{C}$ then
$$r \text{ contains an eigenvalue of } L \implies \det(M - rI) \ni 0.$$
3. Smooth one-parameter family, $\mu \mapsto L_\mu$, with
$$L_1 = M(\ni L) \text{ and } L_0 = D \text{ (diag)},$$
4. Choose discs $D_1, \ldots, D_{m+1}$ with
$$D_k \ni \lambda_k \text{ for } k \leqslant m, \quad D_{m+1} \ni \lambda_k \; \forall k > m.$$
5. Prove, $\forall \mu \in [0,1], \forall \lambda \in \Gamma_k := \partial D_k$, that
$$\det(L_\mu - \lambda I) \neq 0.$$

The eigenproblem for a general linear operator $L \in \mathscr{B}(\mathcal{A}, \mathcal{A})$

Firstly, establish the structure of the spectrum (incl. multiplicities) and gain crude bounds via 'contracted matrices'.

We want to bound eigenfunction-eigenvalue pairs (V, λ) with

$$(L - \lambda I)V = 0.$$

Take a linear (coordinate) functional φ. Normalise V so that

$$\lambda = \varphi(V).$$

Now solve the (nonlinear in V) problem: either

$$\big(L - \varphi(V)I\big)V = 0, \quad \text{or} \quad \begin{cases} \psi(V) - 1 = 0 \\ (L - \lambda I)V = 0 \quad \text{for } (\lambda, V). \end{cases}$$

(for some chosen normalisation functional ψ) again by proving a Newton-like op is a contraction map.

Detail: Newton's method for eigenfunctions

We want

$$0 = F(V) := DT(G^*)V - \varphi(V)V.$$

Frechet derivative $DF(V)$ given formally by

$$DF(V) : \delta V \mapsto DT(G^*)\delta V - \varphi(\delta V)V - \varphi(V)\delta V.$$

Take a fixed invertible linear operator $\Lambda \simeq DF(V^0)$ and form Newton-like operator (Care: dependency problem.)

$$\Psi : V \mapsto V - \Lambda\left[DT(G^*)V - \varphi(V)V\right].$$

Bound $\|\Psi(V^0) - V^0\| \leqslant \hat{\varepsilon}$ and $\|D\Psi(V)\delta V\| \leqslant \hat{\kappa} < 1$
$\forall\, G \in B(G^0; 0, \rho)$ and $\forall\, V \in B(V^0; 0, \hat{\rho})$ and confirm $\hat{\varepsilon} < \hat{\rho}(1 - \hat{\kappa})$.

Example ($d = 4$): eigenfunction V corresponding to δ_4 yields

$$\delta_4 = \varphi(V) = +7.\,2846862170\ 7334336430\ \ldots$$

Eigenfunction controlling scaling of added noise

Modify iteration $x_{n+1} = f_\mu(x_n)$, to add i.i.d. noise

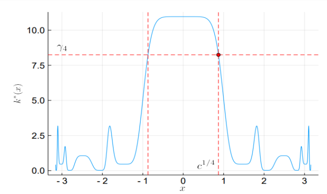
$$x_{n+1} = f_\mu(x_n) + \varepsilon \xi_n.$$

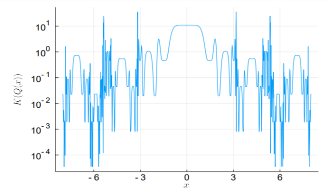
Examine $R(f)$ and take $f \to g^*$ and $\varepsilon \to 0$ to give eigenproblem

$$\gamma^2 W = \mathcal{L} W := {L_1}^2 \cdot W(Q(G(Q(a)X))) + {L_2}^2 \cdot W(Q(a)X),$$

with $L_1 := a^{-1}$, $L_2 := a^{-1} G'(Q(G(Q(a)X))) \cdot Q'(G(Q(a)X))$.

Example ($d = 4$): Eigenfunction $w = W \circ Q$ ('redistribute noise')





Eigenvalue ('scale var'): $\gamma_4 = +8.24391\,08542\,52586\,81839\ldots$

Detail: Newton method for noise eigenproblem

Define $\gamma = \varphi(W)$ and rewrite as

$$\mathcal{F}(W) := \left(\mathcal{L} - \varphi(W)^2 I\right) W = 0.$$

The operator $\mathcal{F}$ has Frechet derivative

$$D\mathcal{F}(W) : \delta W \mapsto \mathcal{L}\delta W - 2\varphi(W)\varphi(\delta W)W - \varphi(W)^2\delta W.$$

Newton-like operator (Care: dependency problem.)

$$\Theta(W) := W - \Lambda\mathcal{F}(W),$$

where Λ is a fixed invertible linear operator $\Lambda \simeq [D\mathcal{F}(W^0)]^{-1}$.
Its Frechet derivative is

$$\begin{aligned} &D\Theta(W) : \delta W \\ &\mapsto \delta W - \Lambda D\mathcal{F}(W)\delta W \\ &= \delta W - \Lambda\left[\mathcal{L}\delta W - 2\varphi(W)\varphi(\delta W)W - \varphi(W)^2\delta W\right]. \end{aligned}$$

Detail: Dependency problem for noise operator

Define $\gamma = \varphi(W)$ and rewrite as

$$\mathcal{F}(W) := \left(\mathcal{L} - \varphi(W)^2 I\right) W = 0.$$

Newton-like operator

$$\Theta(W) := W - \Lambda \mathcal{F}(W),$$

To choose Λ, we take

$$D\mathcal{F}(W) \simeq \mathcal{L} - 2\varphi(W^0)W^0 e_0^* - \varphi(W^0)^2 I,$$

where $HW^0 = 0$. Take Λ to be the inverse, which gives

$$\Lambda \delta W_H = -\frac{1}{\varphi(W^0)^2}\delta W_H \quad \text{for } \delta W_H \in H\mathcal{A}.$$

Mitigate the corresponding dependency problem by noting that

$$D\Theta(W)\delta W_H = \left[1 - \left(\frac{\varphi(W)}{\varphi(W^0)}\right)^2\right]\delta W_H - \Lambda \mathcal{L} \delta W_H.$$

Improving the bounds

Improving the bounds

2 independent implementations of the function ball algebra:

(1) Prototype:

`Python` (Decimal multi-precision floating point, rigorous directed rounding: `decimal`, IEEE754. Closures.)

(2) High-performance:

`Julia` (Binary multi-precision, rigorous directed rounding: `BigFloat`, IEEE754. Multiple dispatch. Closures.)

Verification:

Suite of unit tests and functional tests (incl. 'semi-symbolic').

Process-based parallelism protects hardware rounding modes.

Improving the bounds

Example ($d = 4$). Function ball radii achieved:

N	ρ (for G^*)	$\hat{\rho}$ (for V^*)	$\tilde{\rho}$ (for W^*)
40	10^{-20}	10^{-15}	10^{-15}
80	10^{-41}	10^{-36}	10^{-36}
160	10^{-83}	10^{-78}	10^{-77}
320	10^{-165}	10^{-159}	10^{-159}
640	10^{-331}	10^{-325}	10^{-323}

Individual power-series coefficents share (at least) this accuracy.

`Julia` implementation: (1) IEEE multi-precision arithmetic with directed rounding, (2) multi-processing, and (3) closures.

Bounds on Universal constants

E.g., Digits **proven correct** for δ_4 (similarly for a_4, α_4, γ_4):

$$\delta_4 = \varphi(V^*)$$

$\delta_4 =$	$+7.$	2846862170	7334336430	8930567995	5530694780
		4661979979	0659072121	2901883462	1435067620
		0657264503	1360371147	0784357866	9255573693
		3221121594	9170167056	0272610414	2834709598
		2287873290	2387885867	2064166568	1895073101
		1658106317	3127916581	6323366267	7746542527
		7844194832	0362437902	4983698686	8146702404
		9663158059	7051641021	9527093166	3172744588
		9929...			(325 d)

Coupled systems: Doubling operator

Acting on pairs of maps of two variables:

$$R : \begin{pmatrix} g(x,y) \\ f(x,y) \end{pmatrix} \mapsto \begin{pmatrix} \alpha g\big(g(x/\alpha, y/\beta), f(x/\alpha, y/\beta)\big) \\ \beta f\big(g(x/\alpha, y/\beta), f(x/\alpha, y/\beta)\big) \end{pmatrix},$$

in which we define

$$\begin{aligned} \alpha &:= g\big(g(0,0), f(0,0)\big)^{-1}, \\ \beta &:= f\big(g(0,0), f(0,0)\big)^{-1}, \end{aligned}$$

which preserves the normalisation $g(0,0) = 1$ and $f(0,0) = 1$.

A number of interesting fixed points (e.g., FS-type, Bicritical) and periodic orbits (e.g., C-type).

Ansatz for a particular form of unidirectional coupling

Ansatz (relevant to "FS-type" universality)

$$\begin{aligned} g(x,y) &:= \tilde{g}(x), \\ f(x,y) &:= y + \tilde{f}(x). \end{aligned}$$

Gives

$$R : \begin{pmatrix} \tilde{g}(x) \\ y + \tilde{f}(x) \end{pmatrix} \mapsto \begin{pmatrix} \alpha \tilde{g}\big(\tilde{g}(x/\alpha)\big) \\ y + \beta\Big(\tilde{f}(x/\alpha) + \tilde{f}\big(\tilde{g}(x/\alpha)\big)\Big) \end{pmatrix},$$

with

$$\begin{aligned} \alpha &:= \tilde{g}\big(\tilde{g}(0)\big)^{-1}, \\ \beta &:= \Big(\tilde{f}(0) + \tilde{f}\big(\tilde{g}(0)\big)\Big)^{-1}, \end{aligned}$$

which preserves the normalisation $\tilde{g}(0) = 1$ and $\tilde{f}(0) = 1$.

Conjectured fixed point

Seek nontrivial solutions, $\tilde{f}$, of the functional equation

$$y + \tilde{f}(x) \quad = \quad y + \beta\Big(\tilde{f}(x/\alpha) + \tilde{f}\big(g^*(x/\alpha)\big)\Big).$$

[Kuznetsov et al.] For $d = 2$ nontrivial solutions include:
(1) $\tilde{f}(x) = g^*(x) - x$ and $\beta = \alpha = -2.50290787\ldots$.
(2) A conjectured (even) solution for which $\beta \simeq -4.58619671$.

Proof (2): Seek nontrivial fixed point $\tilde{f}$ of

$$\mathcal{K} : \tilde{f}(x) \quad \mapsto \quad \beta\Big(\tilde{f}\,(x/\alpha) + \tilde{f}\big(g^*\,(x/\alpha)\big)\Big).$$

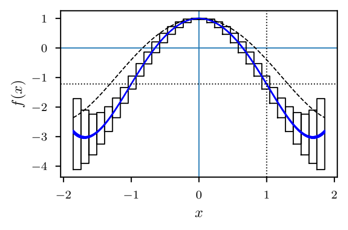
Write $\tilde{f}(x) = \tilde{F}(x^2)$ and $X = x^2$, use $B(G^0, \rho_G) \ni G^*$, and consider

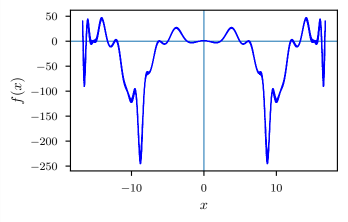
$$K : \tilde{F}(X) \quad \mapsto \quad \beta\Big(\tilde{F}(X/\alpha^2) + \tilde{F}\big(G^*(X/\alpha^2)^2\big)\Big),$$

where $\quad \alpha := G^*(G^*(0)^2)^{-1}, \quad \beta := \big(\tilde{F}(0) + \tilde{F}(G^*(0)^2)\big)^{-1}.$

Proof of existence for the fixed-point functions $\tilde{F}, \tilde{f}$ ($d = 2$)

Can take $N = 20$ with $\rho_G = 10^{-11}$ giving $\varepsilon_F = 1.6 \times 10^{-8}$. Then $\rho_F = 10^{-7}$ gives $\kappa_F = 4.2 \times 10^{-2}$, $\beta \in [-4.58620, -4.58619]$.



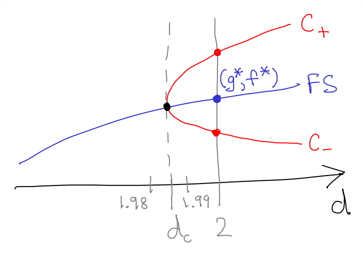


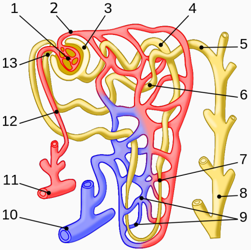
With $N = 160$, $\rho_G = 10^{-98}$, $\rho_F = 10^{-93}$: $\kappa_F = 1.7 \times 10^{-23}$ and

$$\beta \in -4.5861967092\,9064476823\,4114397284\,0585953000\,3383644172\,0205026501\,8619608373\,5020375014\,5765904864 - [3, 4) \times 10^{-91}.$$

Further work

1/ Bifurcation of the FS-type fixed point into the C-type 2-cycle:
$\sigma(DR(g^*, f^*)) \ni \lambda = \delta_2/\beta_2 = -1.018\ldots$ (bidirectional coupling).
Period-doubling in the dynamics of R itself (at $d = d_c < 2$):





2/ Bicritical (B-type) fixed point.

[Kuznetsov, Sataev (1992). Kuznetsov, Kuznetsov, Sataev (2005). Kuznetsov, Mailybaev, Sataev (2008). Laugesen et. al. (2011)]

Other scenarios: example

Rigid complex rotation

$$z \mapsto \lambda z, \quad |\lambda| = 1.$$

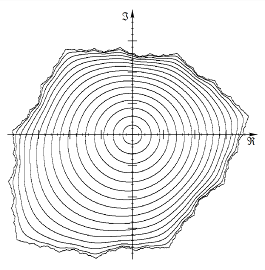
Perturb (for ω golden mean):

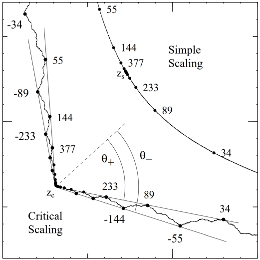
$$f : z \mapsto e^{2\pi\omega i} z + \sum_{k=2}^{\infty} f_k z^k.$$

Critical scaling at Fibonacci iterates

$$R : \begin{pmatrix} F \\ G \end{pmatrix} \mapsto \alpha \begin{pmatrix} G \\ F \circ G \end{pmatrix} \alpha^{-1}.$$

[Stirnemann 1992, 1993; Burbanks, Osbaldestin, Stirnemann 1997, 1998, 1999]





Some computational issues and challenges

- Axioms and unit tests.
- Dependency problems.
- Directed rounding.
- Thread safety / multiprocessing / memory management.
- Efficient rigorous arithmetic (e.g., reducing allocations).
- Alternative interval / complex representations.
- Finding good domains.
- Automation of Frechet differentiation.
- Efficient spectral bounds.
- Adaptive coverings / Adaptive precision.
- Alternative bases.
- Alternative norms / spaces.
- Multivariable framework.
- Generic framework.

Thanks and References

[1] Burbanks, Osbaldestin. *Existence of the FS-type renormalisation fixed point for unidirectionally-coupled pairs of maps*, J. Phys. A: Math. Theor. 56, 195202 (2023)

[2] Burbanks, Osbaldestin, Thurlby. *Rigorous computer-assisted bounds on the period doubling renormalization fixed point and eigenfunctions in maps with critical point of degree 4*, J. Math. Phys. 62, 112701 (2021)

[3] Burbanks, Osbaldestin, Thurlby. *Rigorous computer-assisted bounds on renormalisation fixed point functions, eigenfunctions, and universal constants*, arXiv: 2103.05991 [math.DS] (2021)

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Additional materials

Detail: Formal Fréchet derivative $DR(g,f)$

Let $\tilde{g} = g(x/\alpha, y/\beta)$, $\tilde{f} = f(x/\alpha, y/\beta)$, then

$$DR(g,f) : (\delta g(x,y), \delta f(x,y)) \mapsto$$
$$\begin{pmatrix} \delta\alpha \cdot g(\tilde{g},\tilde{f}) + \alpha\big(\delta g(\tilde{g},\tilde{f}) + \partial_1 g(\tilde{g},\tilde{f}) \cdot \delta\tilde{g} + \partial_2 g(\tilde{g},\tilde{f}) \cdot \delta\tilde{f}\big) \\ \delta\beta \cdot f(\tilde{g},\tilde{f}) + \beta\big(\delta f(\tilde{g},\tilde{f}) + \partial_1 f(\tilde{g},\tilde{f}) \cdot \delta\tilde{g} + \partial_2 f(\tilde{g},\tilde{f}) \cdot \delta\tilde{f}\big) \end{pmatrix},$$

where the perturbations $\delta\tilde{g}$, $\delta\tilde{f}$ are

$$\begin{aligned}
\delta\tilde{g} &= \delta g(x/\alpha, y/\beta) + \partial_1 g(x/\alpha,\ y/\beta) \cdot (-1/\alpha^2)\delta\alpha \cdot x \\
&\quad + \partial_2 g(x/\alpha,\ y/\beta) \cdot (-1/\beta^2)\delta\beta \cdot y, \\
\delta\tilde{f} &= \delta f(x/\alpha, y/\beta) + \partial_1 f(x/\alpha,\ y/\beta) \cdot (-1/\alpha^2)\delta\alpha \cdot x \\
&\quad + \partial_2 f(x/\alpha,\ y/\beta) \cdot (-1/\beta^2)\delta\beta \cdot y, \\
\delta\alpha &= -\alpha^2 \cdot \big(\delta g(g(0,0),\ f(0,0)) + \partial_1 g(g(0,0),\ f(0,0)) \cdot \delta g(0,0) \\
&\quad + \partial_2 g(g(0,0),\ f(0,0)) \cdot \delta f(0,0)\big), \\
\delta\beta &= -\beta^2 \cdot \big(\delta f(g(0,0),\ f(0,0)) + \partial_1 f(g(0,0),\ f(0,0)) \cdot \delta g(0,0) \\
&\quad + \partial_2 f(g(0,0),\ f(0,0)) \cdot \delta f(0,0)\big).
\end{aligned}$$

Detail: Improving the bounds (FS-type fixed point)

Function ball radii achieved (using truncation N, precision P):

N	P	ρ_G	ε_F	ρ_F	κ_F
20	20	10^{-11}	1.6×10^{-8}	10^{-7}	4.2×10^{-2}
40	40	10^{-23}	2.6×10^{-20}	10^{-19}	5.5×10^{-5}
80	80	10^{-48}	4.8×10^{-45}	10^{-43}	4.7×10^{-11}
160	160	10^{-98}	9.1×10^{-95}	10^{-93}	1.7×10^{-23}

Implementation: (1) IEEE multi-precision arithmetic & directed rounding, (2) ~~threads~~ multi-processing, and (3) closures.

2 independent implementations of the function-ball algebra:

(a) Prototype: `Python` (decimal).
(b) High-performance: `Julia` (binary). Multiple dispatch.

Verification: Over 1200 unit tests and functional tests.

Detail: Dealing with R directly

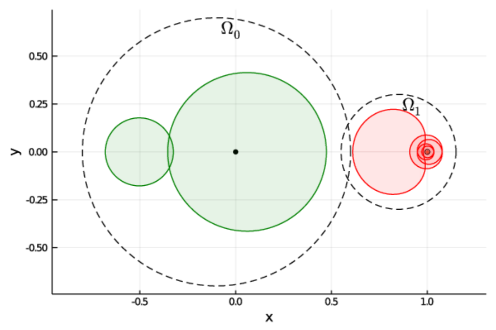
Banach space, $\mathscr{A}(\Omega_0) \times \mathscr{A}(\Omega_1)$, of pairs, $g = g_0 \oplus g_1$, on domain $\Omega = \Omega_0 \cup \Omega_1$ (representing hybrid functions), where $0 \in \Omega_0 = \mathbb{D}(c_0, r_0)$, $1 \in \Omega_1 = \mathbb{D}(c_1, r_1)$, and $\Omega_0 \cap \Omega_1 \neq \emptyset$,

$$\|g\| := \|g_0\| + \|g_1\|.$$

Example ($d = 4$):
$\Omega_0 = \mathbb{D}(-0.1, 0.7)$,
$\Omega_1 = \mathbb{D}(0.85, 0.3)$.

Domain extension:
$R(g) = a^{-1} g(g(ax))$,

$$a\overline{\Omega} \subset \Omega_0,$$
$$g(a\overline{\Omega}) \subset \Omega_1.$$



Detail: Spectrum of $DR(g)$ itself (for $d = 4$)

The spectrum of $DR(g^*)$ has 5 eigenvalues with $|\lambda| > 1$:

$$\alpha_4^4, \delta_4, \alpha_4^3, \alpha_4^2, \alpha_4,$$

($DT(G^*)$ has 2 of these: α_4^4, δ_4.)

α_4^4: non-essential (coordinate-change) eigenvalue.
α_4^3, α_4^1: destroy symmetry of quartic critical point.
α_4^2: tricritical vector scaling for locally bimodal maps where one quadratic extremum is mapped to another: fixed point $R(q_2) = q_2$: $q_2(x) = g^*(\sqrt{x})^2 = G^*(x^2)^2$ with scaling $g^*(1)^2 = \alpha_4^2$.

Detail: Contracted vectors

Banach space $X = PX \oplus HX$ with basis $\{e_1, e_2, \ldots, e_m\}$ of PX:

$$Px = \sum_{i=1}^{m} \phi_i(x) e_i.$$

Defn: A 'contracted' interval vector $u = (u_1, \ldots, u_{m+1})$ contains x (written $x \in u$) if

$$\phi_k(x) \in u_k, \quad 1 \leqslant k \leqslant m,$$
$$\text{and} \quad \psi(Hx) \in u_{m+1} \quad \forall \psi \in (HX)^*, \|\psi\| \leqslant 1.$$

Example: If $x \in X$ and intervals $u_k \ni \phi_k(x)$, then

$$x \in u = (u_1, \ldots, u_m, [-\|Hx\|, +\|Hx\|]).$$

Important Properties:

$$x \in u,\ y \in v \implies x + y \in u + v,$$
$$x \in u,\ \lambda \in \ell \implies \lambda x \in \ell u.$$

Detail: Contracted matrices

Let $L \in \mathscr{B}(X, X)$. In block form, w.r.t. $X = PX \oplus HX$ and $\{e_k\}$:

$$L = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & \theta_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} & \theta_m \\ \hline t_1 & \cdots & t_m & \Theta \end{array}\right).$$

If the 'matrix elements' $a_{jk} \in c_{jk}$, then L is contained by

$$C = \left(\begin{array}{ccc|c} c_{11} & \cdots & c_{1m} & [-\|\theta_1\|, +\|\theta_1\|] \\ \vdots & \ddots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mm} & [-\|\theta_m\|, +\|\theta_m\|] \\ \hline [-\|t_1\|, +\|t_1\|] & \cdots & [-\|t_m\|, +\|t_m\|] & [-\|\Theta\|, +\|\Theta\|] \end{array}\right)$$

Important Properties:

$$L \in C,\ x \in u \implies Lx \in Cu,$$

$$L \in C,\ \lambda \in \sigma(L),\ \lambda \in r \implies \det(C - rI) \ni 0.$$

Detail: Delta eigenfunction, choice of fixed linear operator

Newton-like operator for $F(V) = (DT(G^*) - \varphi(V))V = 0$:

$$\Psi : V \mapsto V - \Lambda\left[DT(G^*)V - \varphi(V)V\right].$$

Aim: bound $\|\Psi(V^0) - V^0\| \leqslant \widehat{\epsilon}$ via ops on (singleton) ball $B(V^0; 0, 0)$, bound $\|D\Psi(V)(e_j)\| \leqslant \kappa < 1$ for all $V \in B(V^0; 0, \widehat{\rho})$ and all $j \geqslant 0$.

Anticipating a dependency problem, examine $DF(V)$:

$$DF(V)\delta V = (DT(G) - Ve_k^* - V_k I)\,\delta V,$$

$$DF(V) \simeq \Delta - V^0 e_k^* - V_k^0 I,$$

Assume wlog, $k = 0$ so that $\varphi(V) = V_0$ then, for $HV^0 = 0$, take

$$\Gamma = \Delta - V^0 e_0^* - V_0^0 I = \left(\begin{array}{cccc|c} \Delta_{00} - 2V_0^0 & \Delta_{01} & \cdots & \Delta_{0N} & 0 \\ \Delta_{10} - V_1^0 & \Delta_{11} - V_0^0 & \cdots & \Delta_{1N} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{N0} - V_N^0 & \Delta_{N1} & \cdots & \Delta_{NN} - V_0^0 & 0 \\ \hline 0 & 0 & \cdots & 0 & -V_0^0 I \end{array}\right).$$

Detail: Dependency problem for delta eigenfunction

Recall that

$$\Psi : V \mapsto V - \Lambda\big[DT(G)V - \varphi(V)V\big],$$

with Frechet derivative

$$D\Psi(V) : \delta V \mapsto \delta V - \Lambda\big[DT(G)\delta V - \varphi(\delta V)V - \varphi(V)\delta V\big].$$

Consider the action of $D\Psi(V)$ on $\delta V_H \in H\mathcal{A}$:

$$\begin{aligned} D\Psi(V)\delta V_H &= \delta V_H - \Lambda\big[DT(G)\delta V_H - \varphi(V)\delta V_H\big] \quad (*) \\ &= \left[1 - \frac{\varphi(V)}{\varphi(V^0)}\right]\delta V_H - \Lambda DT(G)\delta V_H, \quad (**) \end{aligned}$$

since $\varphi(\delta V_H) = 0$ and the action of Λ on the high-order part of the space is given by $-(1/V_0^0)I$. Note: for V close to V^0, the first term in $(**)$ is close to zero.

When bounding $\|D\Psi(B(V^0; 0, \hat{\rho}))(E_H)\|$, we therefore use $(**)$ for $D\Psi(V)\delta V_H$, with $V \in B(V^0; 0, \widehat{\rho})$, when computing $D\Psi(V)E_H$.