Critical points of the multipliers

Igors Gorbovickis

Constructor University
(formerly: Jacobs University, Bremen)

March 11, 2024
The Filled-in Julia sets

- Iteration of rational maps on the Riemann sphere – first studied by *Pierre Fatou* and *Gaston Julia* in the late 1910s.
- The subject gained increased attention again in the 1980s, partly due to the emergence of computer-generated images.
The Filled-in Julia sets

- Iteration of rational maps on the Riemann sphere – first studied by Pierre Fatou and Gaston Julia in the late 1910s.
- The subject gained increased attention again in the 1980s, partly due to the emergence of computer-generated images.
- Quadratic family: \( \{ f_c(z) = z^2 + c \mid c \in \mathbb{C} \} \).
The Filled-in Julia sets

- Iteration of rational maps on the Riemann sphere – first studied by Pierre Fatou and Gaston Julia in the late 1910s.
- The subject gained increased attention again in the 1980s, partly due to the emergence of computer-generated images.
- Quadratic family: \( \{ f_c(z) = z^2 + c \mid c \in \mathbb{C} \} \).
- Filled-in Julia set of \( f_c \):

\[
K_c := \{ z \in \mathbb{C} \mid \limsup_{k \to +\infty} |f_c^k(z)| < \infty \}.
\]
The Mandelbrot set $\mathbb{M}$

- The Mandelbrot set: $\mathbb{M} := \{ c \in \mathbb{C} \mid 0 \in K_c \}$. $\mathbb{M}$ is the set of all parameters $c \in \mathbb{C}$, for which $K_c$ is connected.
The Mandelbrot set $\mathcal{M}$

- The Mandelbrot set: $\mathcal{M} := \{ c \in \mathbb{C} \mid 0 \in K_c \}$. 
  $\mathcal{M}$ is the set of all parameters $c \in \mathbb{C}$, for which $K_c$ is connected.

- $\mathcal{O} = \langle z_0, z_1, \ldots, z_{k-1} \rangle$ is a periodic orbit of period $k$ for $f_c$.
- Multiplier of $\mathcal{O}$: $\rho_\mathcal{O}(c) = f'_c(z_0)f'_c(z_1)\ldots f'_c(z_{k-1})$.
- A quadratic polynomial $f_c$ is hyperbolic iff it has a periodic orbit $\mathcal{O}$ with $|\rho_\mathcal{O}(c)| < 1$. (attracting periodic orbit)
The Mandelbrot set $\mathbb{M}$

- The Mandelbrot set: $\mathbb{M} := \{c \in \mathbb{C} \mid 0 \in K_c\}$. $\mathbb{M}$ is the set of all parameters $c \in \mathbb{C}$, for which $K_c$ is connected.

- $\mathcal{O} = \langle z_0, z_1, \ldots, z_{k-1} \rangle$ is a periodic orbit of period $k$ for $f_c$.
- Multiplier of $\mathcal{O}$: $\rho_{\mathcal{O}}(c) = f'_c(z_0)f'_c(z_1)\ldots f'_c(z_{k-1})$.
- A quadratic polynomial $f_c$ is hyperbolic iff it has a periodic orbit $\mathcal{O}$ with $|\rho_{\mathcal{O}}(c)| < 1$. (attracting periodic orbit)
- *Density of hyperbolicity conjecture*: every connected component of the interior of $\mathbb{M}$ is a hyperbolic component.
Theorem (Sullivan, Douady-Hubbard): The multiplier $\rho_O$ of an attracting periodic orbit is a Riemann mapping of the corresponding hyperbolic component $H$.

$$\rho_O^{-1} : \mathbb{D} \to H$$ is a conformal isomorphism.

Observation: If $\rho_O^{-1}$ can be extended univalently to a fixed neighborhood $U \ni \mathbb{D}$, then Koebe Distortion Theorem provides bounds on the shape of $H$. 
Theorem (Sullivan, Douady-Hubbard): The multiplier $\rho_O$ of an attracting periodic orbit is a Riemann mapping of the corresponding hyperbolic component $H$.

$$\rho_O^{-1} : \mathbb{D} \to H$$  

is a conformal isomorphism.

Observation: If $\rho_O^{-1}$ can be extended univalently to a fixed neighborhood $U \supset \mathbb{D}$, then Koebe Distortion Theorem provides bounds on the shape of $H$.

Critical values of $\rho_O$ are the only obstacles for an analytic extension of $\rho_O^{-1}$ beyond $\mathbb{D}$.

Problem: Study critical points (and critical values) of the multiplier maps $\rho_O$. (i.e., the parameters $c \in \mathbb{C}$, such that $\rho'_O(c) = 0$.)
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 3$

Number of critical points $= 2$
How to compute critical points? (joint with A. Belova)

\[
\frac{d\rho_k}{dc} = 2^k \left[ z' \prod_{i=1}^{k-1} f_c^{\circ i}(z) + z \sum_{i=1}^{k-1} \left( \frac{df_c^{\circ i}(z)}{dc} \prod_{j=1}^{k-1} f_c^{\circ j}(z) \right) \right]
\]

\[
= 2^k \left( \sum_{i=0}^{k-1} z_i' \prod_{0 \leq j < k, j \neq i} z_j \right),
\]

where \( z \) is a periodic point of period \( k \), \( \rho_k \) is its multiplier, and

\[
z' = \frac{dz}{dc} = \frac{\partial f_c^{\circ k}}{\partial c}(z) (1 - \rho_k(c))^{-1}.
\]
How to compute critical points? (joint with A. Belova)

\[
\frac{d \rho_k}{dc} = 2^k \left[ z' \prod_{i=1}^{k-1} f_c^{\circ i}(z) + z \sum_{i=1}^{k-1} \left( \frac{df_c^{\circ i}(z)}{dc} \prod_{j=1}^{k-1} f_c^{\circ j}(z) \right) \right]
\]

\[
= 2^k \left( \sum_{i=0}^{k-1} z_i' \prod_{0 \leq j < k, j \neq i} z_j \right),
\]

where \( z \) is a periodic point of period \( k \), \( \rho_k \) is its multiplier, and

\[
z' = \frac{dz}{dc} = \frac{\partial f_c^{\circ k}}{\partial c}(z) \left( 1 - \rho_k(c) \right)^{-1}.
\]

Idea: use the 3-dimensional Newton’s method for the system

\[
\begin{cases}
    f_c^{\circ k}(z) - z = 0 \\
    z' - \frac{\partial f_c^{\circ k}}{\partial c}(z) \left( 1 - \frac{\partial f_c^{\circ k}}{\partial z}(z) \right)^{-1} = 0 \\
    \frac{d \rho_k}{dc} = 0,
\end{cases}
\]

with three unknowns \( c, z, z' \).
Critical points of the multiplier maps $\rho_\mathcal{O}$, $|\mathcal{O}| = 4$

Number of critical points = 6
Critical points of the multiplier maps $\rho_\mathcal{O}$, $|\mathcal{O}| = 5$

Number of critical points = 20
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 6$

Number of critical points = 38
When is \( c = 0 \) a critical point of \( \rho_\mathcal{O} \)?

When \( c = 0 \), and \( \mathcal{O} = \langle z_0, \ldots, z_{k-1} \rangle \) is a periodic orbit of \( f_0 \),

\[
\rho'_\mathcal{O}(0) = -2^k \sum_{j=0}^{k-1} z_j^{-1}.
\]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( z_0 )</th>
<th>( 6 )</th>
<th>( 12 )</th>
<th>( 18 )</th>
<th>( 20 )</th>
<th>( 21 )</th>
<th>( 24 )</th>
<th>( 30 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 6 )</td>
<td>( e^{2\pi i/9} )</td>
<td>( e^{2\pi i/45} )</td>
<td>( e^{2\pi i/27} )</td>
<td>( e^{2\pi i/25} )</td>
<td>( e^{2\pi i/49} )</td>
<td>( e^{2\pi i/153} )</td>
<td>( e^{2\pi i/99} )</td>
<td></td>
</tr>
</tbody>
</table>

Table: The list of all periods \( k \leq 30 \), for which there exists a multiplier map \( \rho_\mathcal{O} \) with a critical point at \( c = 0 \). (\( z_0 \) is a corresponding periodic point.)

**Lemma:** For every \( k \in \mathbb{N} \), the point \( z_k = \exp(2\pi i/3^{k+1}) \) belongs to a periodic orbit \( \mathcal{O}_k \) of period \( n_k = 2 \cdot 3^k \) for the polynomial \( f_0(z) = z^2 \), and \( \rho'_{\mathcal{O}_k}(0) = 0 \).
Critical points of the multiplier maps $\rho_\mathcal{O}$, $|\mathcal{O}| = 7$

Number of critical points $= 102$
Critical points of the multiplier maps $\rho_\mathcal{O}, \quad |\mathcal{O}| = 8$

Number of critical points $= 198$
For any $s \in \mathbb{C}$ and any $k \in \mathbb{N}$,

$X_{s,k} := \{ c \in \mathbb{C} \mid \rho'_\mathcal{O}(c) = s, \text{ for some periodic orbit } \mathcal{O} \}$. (Points in $X_{s,k}$ are counted with multiplicity.)

$$\nu_{s,k} := \frac{1}{\# X_{s,k}} \sum_{c \in X_{s,k}} \delta_c.$$ 

**Equidistribution Theorem (Firsova, G.):** For every sequence of complex numbers $\{s_k\}_{k \in \mathbb{N}}$, such that

$$\limsup_{k \to +\infty} \frac{1}{k} \log |s_k| \leq \log 2,$$

the sequence of measures $\{\nu_{s_k,k}\}_{k \in \mathbb{N}}$ converges to $\mu_{\text{bif}}$ in the weak sense of measures on $\mathbb{C}$, as $k \to \infty$. 
Theorem (Levin 1989, Bassanelli-Berteloot 2011, Buff-Gauthier 2015): For any $\rho_0 \in \mathbb{C}$, the set of parameters $c$ (counted with multiplicity), such that $\rho_{\mathcal{O}}(c) = \rho_0$, for some $\mathcal{O}$ of period $k$, equidistributes on the boundary of $\mathbb{M}$, as $k \to \infty$. 

$\text{X} \subset \mathbb{C}$ is the accumulation set of critical points of the multipliers. 

Theorem (Firsova, G.): The accumulation set $X$ is bounded, connected and contains the Mandelbrot set $\mathbb{M}$. Furthermore, the set $X \setminus \mathbb{M}$ is nonempty and has a nonempty interior, and every critical point of any multiplier is in $X$. 

Igors Gorbovickis

Critical points of the multipliers
Theorem (Levin 1989, Bassanelli-Berteloot 2011, Buff-Gauthier 2015): For any $\rho_0 \in \mathbb{C}$, the set of parameters $c$ (counted with multiplicity), such that $\rho_O(c) = \rho_0$, for some $O$ of period $k$, equidistributes on the boundary of $\mathbb{M}$, as $k \to \infty$.

$X \subset \mathbb{C}$ is the accumulation set of critical points of the multipliers.

Theorem (Firsova, G.): The accumulation set $X$ is bounded, connected and contains the Mandelbrot set $\mathbb{M}$. Furthermore, the set $X \setminus \mathbb{M}$ is nonempty and has a nonempty interior, and every critical point of any multiplier is in $X$. 
The accumulation set $\mathcal{X}$
Equidistribution: Idea of the proof

**Step 1:** For each measure $\nu_k$, construct a potential (a subharmonic function) $u_k : \mathbb{C} \to [-\infty, +\infty)$, such that

$$\Delta u_k = \nu_k.$$

**Step 2:** Then convergence $u_k \to G_M$ in $L^1_{\text{loc}}$ as $k \to \infty$ implies weak convergence of measures $\nu_k \to \mu_{\text{bif}}$. 

Lemma (Buff, Gauthier.): Any subharmonic function $u : \mathbb{C} \to [-\infty, +\infty)$ which coincides with $G_M$ outside $M$, coincides with $G_M$ everywhere.
Equidistribution: Idea of the proof

Step 1: For each measure $\nu_k$, construct a potential (a subharmonic function) $u_k : \mathbb{C} \rightarrow [-\infty, +\infty)$, such that

$$\Delta u_k = \nu_k.$$ 

Step 2: Then convergence $u_k \rightarrow G_M$ in $L^1_{\text{loc}}$ as $k \rightarrow \infty$ implies weak convergence of measures $\nu_k \rightarrow \mu_{\text{bif}}$.

Lemma (Buff, Gauthier.): Any subharmonic function $u : \mathbb{C} \rightarrow [-\infty, +\infty)$ which coincides with $G_M$ outside $M$, coincides with $G_M$ everywhere.
\[ \tilde{S}_k(c, s) := \prod_{\mathcal{O} | (c, \mathcal{O}) \in \mathcal{P}_k} (s - \rho'_k(f_c, \mathcal{O})) \]

\( \tilde{S}_k \) is a rational map in \( c \) with simple poles at primitive parabolic \( c \).

\[ C_k(c) := \prod_{\tilde{c} \in \tilde{\mathcal{P}}_k} (c - \tilde{c}). \]

\[ S_k(c, s) = C_k(c) \tilde{S}_k(c, s) \quad \text{– polynomials in } c \text{ and } s. \]

**Lemma:** \( S_k(c, s) = 0 \), iff \( \rho'_k(f_c, \mathcal{O}) = s \), for some \( k \)-cycle \( \mathcal{O} \) of \( f_c \).
\[ \tilde{S}_k(c, s) := \prod_{O|(c, O) \in P_k} (s - \rho'_k(f_c, O)) \]

\( \tilde{S}_k \) is a rational map in \( c \) with simple poles at primitive parabolic \( c \).

\[ C_k(c) := \prod_{\tilde{c} \in \tilde{P}_k} (c - \tilde{c}). \]

\[ S_k(c, s) = C_k(c) \tilde{S}_k(c, s) \quad - \text{polynomials in} \ c \ \text{and} \ s. \]

**Lemma:** \( S_k(c, s) = 0 \), iff \( \rho'_k(f_c, O) = s \), for some \( k \)-cycle \( O \) of \( f_c \).

For all \( s \in \mathbb{C} \), define the potentials

\[ u_{s,k}(c) := \frac{1}{\deg_c S_k} \log |S_k(c, s)| = \frac{1}{\deg_c S_k} \left[ \log |\tilde{S}_k(c, s)| + \log |C_k(c)| \right]. \]
The root of the multiplier of a periodic orbit $O$:

$$g_O(c) := [\rho_O(c)]^{1/|O|}.$$

$g_O$ is holomorphic on the double-cover of $\mathbb{C} \setminus M$. The family $\{g_O\}$ is normal.
The root of the multiplier of a periodic orbit $O$:
\[ g_O(c) := [\rho_O(c)]^{1/|O|}. \]

$g_O$ is holomorphic on the double-cover of $\mathbb{C} \setminus \mathbb{M}$. The family $\{g_O\}$ is normal.

$\textbf{Lemma}$: For any $\delta > 0$ and a compact subset $K \subset \mathbb{C} \setminus \mathbb{M}$, the following holds:
\[ \lim_{k \to \infty} \frac{\# \{ O \in \text{Orb}^k : \| g_O - 2\sqrt{\phi_M}\| K < \delta \}}{\# \text{Orb}^k} = 1, \]

where $\phi_M: \mathbb{C} \setminus \mathbb{M} \to \mathbb{C} \setminus \overline{D}$ is the conformal diffeomorphism, taking $(1/4, +\infty)$ to $(1, +\infty)$. 
The sets $\mathcal{Y}_c$

- $\text{Orb}_c^+$ is the set of all repelling periodic orbits of $f_c$.
- For every $\mathcal{O} \in \text{Orb}_c^+$, the function
  \[ \nu_{\mathcal{O}}(c) := \frac{\rho'_{\mathcal{O}}(c)}{|\mathcal{O}| \rho_{\mathcal{O}}(c)} = [\log g_{\mathcal{O}}(c)]' \]
  is defined and analytic around $c = c_0$.
- For each $c \in \mathbb{C}$, we consider the set $\mathcal{Y}_c \subset \mathbb{C}$, defined by
  \[ \mathcal{Y}_c := \{ \nu_{\mathcal{O}}(c) \mid \mathcal{O} \in \text{Orb}_c^+ \} . \]
The sets $\mathcal{Y}_c$

- $\text{Orb}^+_c$ is the set of all repelling periodic orbits of $f_c$.
- For every $\mathcal{O} \in \text{Orb}^+_c$, the function
  \[ \nu_\mathcal{O}(c) := \frac{\rho'_\mathcal{O}(c)}{|\mathcal{O}| \rho_\mathcal{O}(c)} = [\log g_\mathcal{O}(c)]' \]
  is defined and analytic around $c = c_0$.
- For each $c \in \mathbb{C}$, we consider the set $\mathcal{Y}_c \subset \mathbb{C}$, defined by
  \[ \mathcal{Y}_c := \left\{ \nu_\mathcal{O}(c) \mid \mathcal{O} \in \text{Orb}^+_c \right\}. \]

Theorem (Firsova, G.): The following two properties hold:

(i) For every parameter $c \in \mathbb{C} \setminus \{-2\}$, the set $\mathcal{Y}_c$ is convex; for $c = -2$, the set $\mathcal{Y}_{-2}$ is the union of a convex set and the point $-\frac{1}{6}$.

(ii) For every parameter $c \in \mathbb{C} \setminus \mathbb{M}$, the set $\mathcal{Y}_c$ is bounded. A parameter $c \in \mathbb{C} \setminus \mathbb{M}$ belongs to $\mathcal{K}$, if and only if $0 \in \mathcal{Y}_c$. 
Hausdorff dimension function: $\delta(c) := \dim_H(J_c)$

**Theorem (Ruelle):** The function $\delta$ is real-analytic in each hyperbolic component (including the complement of $\mathbb{M}$).

**Theorem (Y. M. He, H. Nie):** *(Version for the quadratic family)* If $c \in \mathbb{C}$ is a hyperbolic parameter and $0 \notin \mathcal{V}_c$, then $c$ is not a critical point of the function $\delta$.

**Corollary:** The Hausdorff dimension function $\delta$ has no critical points in $\mathbb{C} \setminus \mathcal{X}$. 
Proof of (i): Averaging Lemma

**Averaging Lemma**: Let \( O_1, O_2 \) be two distinct non-exceptional repelling periodic orbits of \( f_c \). Then for any \( t \in [0, 1] \), there exists a sequence of periodic orbits \( O_3, O_4, \ldots \) of \( f_c \), such that

\[
g_{O_j} \to g_{O_1}^t g_{O_2}^{1-t}, \quad \text{and} \quad \nu_{O_j} \to t\nu_{O_1} + (1-t)\nu_{O_2}
\]

uniformly on a neighborhood of \( c \) for appropriate branches of the powers.
Then \( F_k(c) \) is the free term of the polynomial \( f_c^\circ k(z) \), hence

\[
F_k(c) = 2^{-2^k} \prod_{m \in \mathbb{N}, m | k} \prod_{\mathcal{O} \in \text{Orb}_c^m} \rho_\mathcal{O}(c),
\]

where the product is taken over all \( m \in \mathbb{N} \), such that \( m \) divides \( k \) and over all periodic orbits \( \mathcal{O} \in \text{Orb}_c^m \).

\[
\frac{F'_k(c)}{kF_k(c)} = \sum_{m \in \mathbb{N}, m | k} \sum_{\mathcal{O} \in \text{Orb}_c^m} \frac{m}{k} \nu_\mathcal{O}(c) \to 0,
\]

as \( k \to \infty \) over an appropriate subsequence, provided that \( c \in \text{int}(\mathcal{M}) \) is not parabolic or critically periodic.

Idea: \( 0 \notin \mathcal{Y}_c \) and Averaging Lemma \( \implies \) no convergence in (2).

\( \nu_{\mathcal{O}_1}(c) \leftarrow \nu_{\mathcal{O}_2}(c) \)