Critical points of the multipliers

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The Filled-in Julia sets

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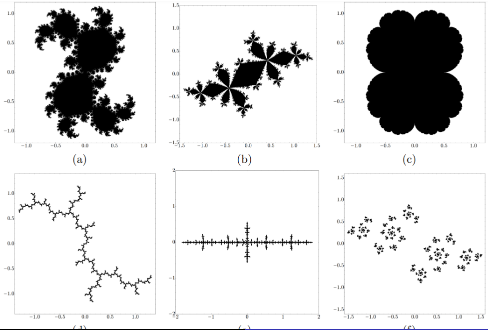
- Quadratic family: $\{ f_c(z) = z^2 + c \mid c \in \mathbb{C} \}$.
- Filled-in Julia set of f_c:

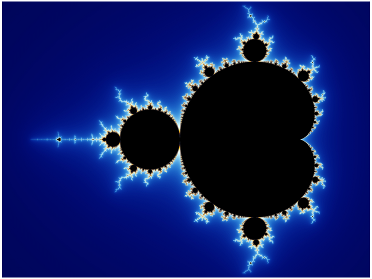
$$K_c := \{z \in \mathbb{C} \mid \limsup_{k\to+\infty} |f_c^{\circ k}(z)| < \infty\}.$$



The Mandelbrot set $\mathbb{M}$

- The Mandelbrot set: $\mathbb{M} := \{c \in \mathbb{C} \mid 0 \in K_c\}$.

$\mathbb{M}$ is the set of all parameters $c \in \mathbb{C}$, for which K_c is connected.



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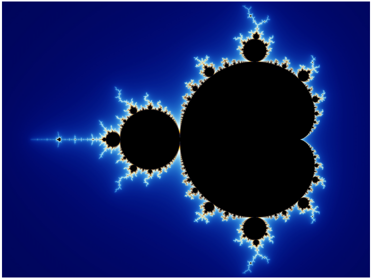

- $\mathcal{O} = \langle z_0, z_1, \dots, z_{k-1} \rangle$ is a periodic orbit of period k for f_c.
- Multiplier of $\mathcal{O}$: $\rho_{\mathcal{O}}(c) = f_c'(z_0) f_c'(z_1) \dots f_c'(z_{k-1})$.
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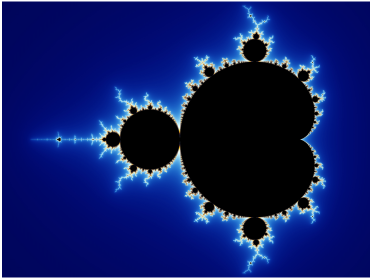

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- A quadratic polynomial f_c is hyperbolic iff it has a periodic orbit $\mathcal{O}$ with $|\rho_{\mathcal{O}}(c)| < 1$. (attracting periodic orbit)
- *Density of hyperbolicity conjecture:* every connected component of the interior of $\mathbb{M}$ is a hyperbolic component.

Multipliers as functions of the parameter

Theorem (Sullivan, Douady-Hubbard): The multiplier $\rho_{\mathcal{O}}$ of an attracting periodic orbit is a Riemann mapping of the corresponding hyperbolic component H.

$$\rho_{\mathcal{O}}^{-1} \colon \mathbb{D} \to H \qquad \text{is a conformal isomorphism.}$$

Observation: If $\rho_{\mathcal{O}}^{-1}$ can be extended univalently to a fixed neighborhood $U \Supset \mathbb{D}$, then Koebe Distortion Theorem provides bounds on the shape of H.

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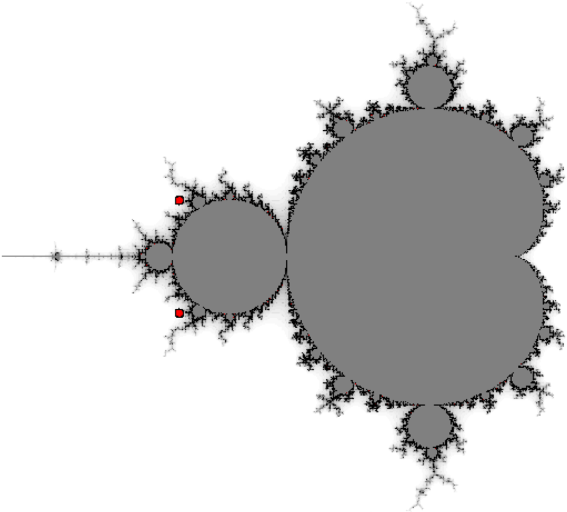
Critical values of $\rho_{\mathcal{O}}$ are the only obstacles for an analytic extension of $\rho_{\mathcal{O}}^{-1}$ beyond $\mathbb{D}$.

Problem: Study critical points (and critical values) of the multiplier maps $\rho_{\mathcal{O}}$. (I.e., the parameters $c \in \mathbb{C}$, such that $\rho'_{\mathcal{O}}(c) = 0$.)

Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 3$

Number of critical points $= 2$



How to compute critical points? (joint with A. Belova)

$$\frac{d\rho_k}{dc} = 2^k \left[z' \prod_{i=1}^{k-1} f_c^{\circ i}(z) + z \sum_{i=1}^{k-1} \left(\frac{df_c^{\circ i}(z)}{dc} \prod_{\substack{j=1 \\ j\neq i}}^{k-1} f_c^{\circ j}(z) \right) \right]$$

$$= 2^k \left(\sum_{i=0}^{k-1} z_i' \prod_{0\le j<k,\, j\neq i}^{k-1} z_j \right),$$

where z is a periodic point of period k, ρ_k is its multiplier, and

$$z' = \frac{dz}{dc} = \frac{\partial f_c^{\circ k}}{\partial c}(z)\,(1-\rho_k(c))^{-1}.$$

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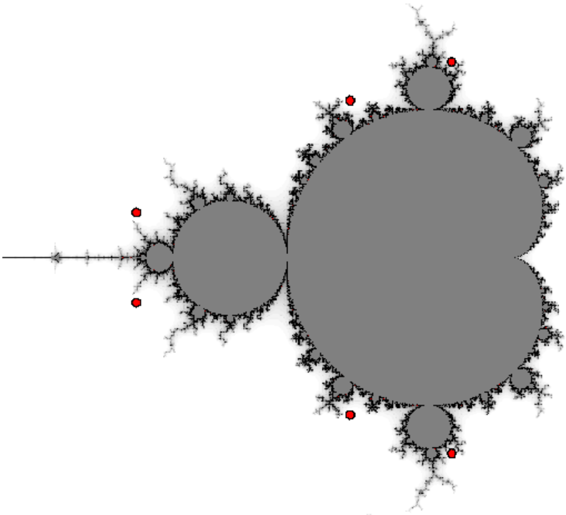
Idea: use the 3-dimensional Newton's method for the system

$$\begin{cases} f_c^{\circ k}(z) - z & = 0 \\ z' - \frac{\partial f_c^{\circ k}}{\partial c}(z)\left(1 - \frac{\partial f_c^{\circ k}}{\partial z}(z)\right)^{-1} & = 0 \\ \frac{d\rho_k}{dc} & = 0, \end{cases} \tag{1}$$

with three unknowns c, z, z'.

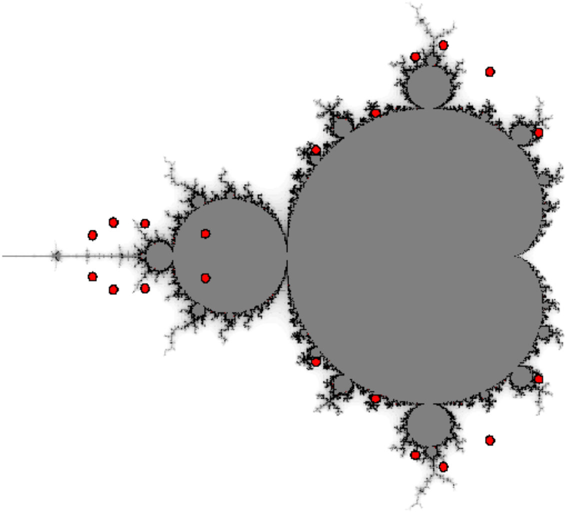
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 4$

Number of critical points $= 6$



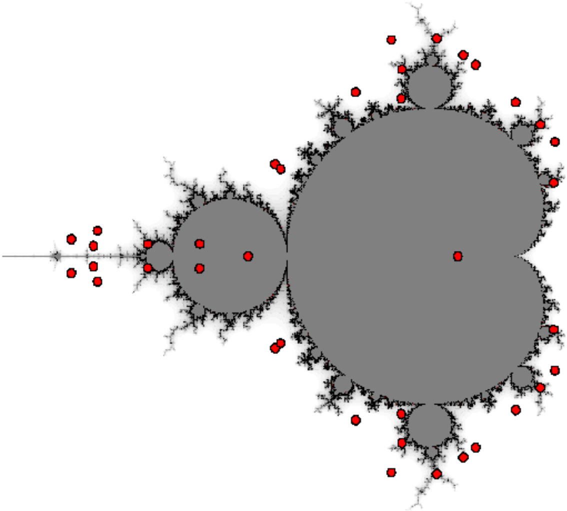
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 5$

Number of critical points = 20



Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 6$

Number of critical points $= 38$



When is $c = 0$ a critical point of $\rho_{\mathcal{O}}$?

When $c = 0$, and $\mathcal{O} = \langle z_0, \dots, z_{k-1} \rangle$ is a periodic orbit of f_0,

$$\rho'_{\mathcal{O}}(0) = -2^k \sum_{j=0}^{k-1} z_j^{-1}.$$

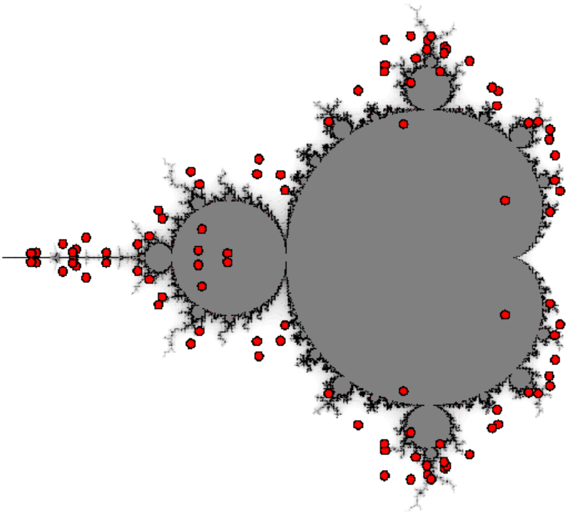
k	6	12	18	20	21	24	30
z_0	$\mathrm{e}^{2\pi i/9}$	$\mathrm{e}^{2\pi i/45}$	$\mathrm{e}^{2\pi i/27}$	$\mathrm{e}^{2\pi i/25}$	$\mathrm{e}^{2\pi i/49}$	$\mathrm{e}^{2\pi i/153}$	$\mathrm{e}^{2\pi i/99}$

Table: The list of all periods $k \le 30$, for which there exists a multiplier map $\rho_{\mathcal{O}}$ with a critical point at $c = 0$. (z_0 is a corresponding periodic point.)

Lemma: For every $k \in \mathbb{N}$, the point $z_k = \exp(2\pi i/3^{k+1})$ belongs to a periodic orbit $\mathcal{O}_k$ of period $n_k = 2 \cdot 3^k$ for the polynomial $f_0(z) = z^2$, and $\rho'_{\mathcal{O}_k}(0) = 0$.

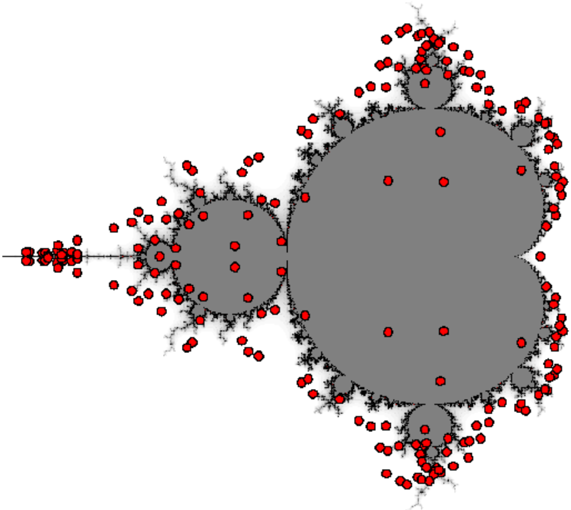
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 7$

Number of critical points = 102



Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 8$

Number of critical points = 198



Equidistribution of critical points of the multipliers

For any $s \in \mathbb{C}$ and any $k \in \mathbb{N}$,

- $X_{s,k} := \{c \in \mathbb{C} \mid \rho'_{\mathcal{O}}(c) = s, \text{ for some periodic orbit } \mathcal{O}\}$. (Points in $X_{s,k}$ are counted with multiplicity.)

$$\nu_{s,k} := \frac{1}{\# X_{s,k}} \sum_{c \in X_{s,k}} \delta_c.$$

Equidistribution Theorem (Firsova, G.): For every sequence of complex numbers $\{s_k\}_{k\in\mathbb{N}}$, such that

$$\limsup_{k\to+\infty} \frac{1}{k} \log |s_k| \leq \log 2,$$

the sequence of measures $\{\nu_{s_k,k}\}_{k\in\mathbb{N}}$ converges to μ_{bif} in the weak sense of measures on $\mathbb{C}$, as $k \to \infty$.

Related results for quadratic polynomials

Theorem (Levin 1989, Bassanelli-Berteloot 2011, Buff-Gauthier 2015): For any $\rho_0 \in \mathbb{C}$, the set of parameters c (counted with multiplicity), such that $\rho_{\mathcal{O}}(c) = \rho_0$, for some $\mathcal{O}$ of period k, equidistributes on the boundary of $\mathbb{M}$, as $k \to \infty$.

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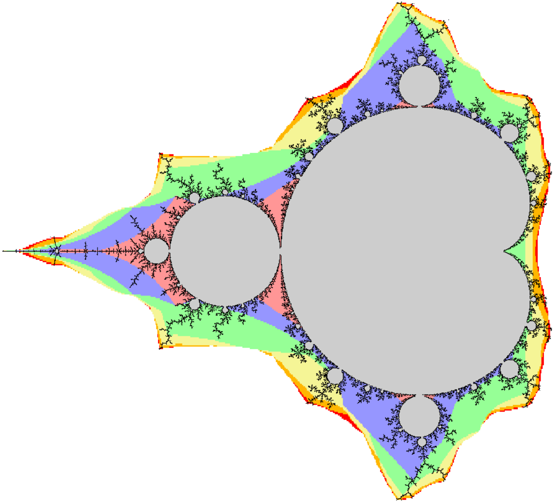
- $\mathcal{X} \subset \mathbb{C}$ is the accumulation set of critical points of the multipliers.

Theorem (Firsova, G.): The accumulation set $\mathcal{X}$ is bounded, connected and contains the Mandelbrot set $\mathbb{M}$. Furthermore, the set $\mathcal{X} \setminus \mathbb{M}$ is nonempty and has a nonempty interior, and every critical point of any multiplier is in $\mathcal{X}$.

The accumulation set $\mathcal{X}$



Equidistribution: Idea of the proof

Step 1: For each measure ν_k, construct a potential (a subharmonic function) $u_k \colon \mathbb{C} \to [-\infty, +\infty)$, such that

$$\Delta u_k = \nu_k.$$

Step 2: Then convergence $u_k \to G_{\mathbb{M}}$ in L^1_{loc} as $k \to \infty$ implies weak convergence of measures $\nu_k \to \mu_{\mathrm{bif}}$.

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Lemma (Buff, Gauthier.): Any subharmonic function $u\colon \mathbb{C} \to [-\infty, +\infty)$ which coincides with $G_{\mathbb{M}}$ outside $\mathbb{M}$, coincides with $G_{\mathbb{M}}$ everywhere.

Potentials

$$\tilde{S}_k(c,s) := \prod_{\mathcal{O}|(c,\mathcal{O})\in P_k} \left(s - \rho'_k(f_c,\mathcal{O})\right)$$

$\tilde{S}_k$ is a rational map in c with simple poles at primitive parabolic c.

$$C_k(c) := \prod_{\tilde{c}\in\tilde{\mathcal{P}}_k} (c - \tilde{c}).$$

$$S_k(c,s) = C_k(c)\tilde{S}_k(c,s) \quad \text{– polynomials in } c \text{ and } s.$$

Lemma: $S_k(c,s) = 0$, iff $\rho'_k(f_c,\mathcal{O}) = s$, for some k-cycle $\mathcal{O}$ of f_c.

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Lemma: $S_k(c,s) = 0$, iff $\rho'_k(f_c,\mathcal{O}) = s$, for some k-cycle $\mathcal{O}$ of f_c.

For all $s \in \mathbb{C}$, define the potentials

$$u_{s,k}(c) := \frac{1}{\deg_c S_k} \log|S_k(c,s)| = \frac{1}{\deg_c S_k}\left[\log|\tilde{S}_k(c,s)| + \log|C_k(c)|\right].$$

Roots of the multiplier maps in $\mathbb{C} \setminus \mathbb{M}$

The root of the multiplier of a periodic orbit $\mathcal{O}$:

$$g_{\mathcal{O}}(c) := [\rho_{\mathcal{O}}(c)]^{1/|\mathcal{O}|}.$$

$g_{\mathcal{O}}$ is holomorphic on the double-cover of $\mathbb{C} \setminus \mathbb{M}$. The family $\{g_{\mathcal{O}}\}$ is normal.

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- Orb^k is the set of all period k cycles of f_c, for $c \in \mathbb{C} \setminus \mathbb{M}$.

Lemma: For any $\delta > 0$ and a compact subset $K \subset \mathbb{C} \setminus \mathbb{M}$, the following holds:

$$\lim_{k \to \infty} \frac{\#\{\mathcal{O} \in Orb^k \colon \|g_{\mathcal{O}} - 2\sqrt{\phi_{\mathbb{M}}}\|_K < \delta\}}{\# Orb^k} = 1,$$

where
$\phi_{\mathbb{M}} \colon \mathbb{C} \setminus \mathbb{M} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is the conformal diffeomorphism, taking $(1/4, +\infty)$ to $(1, +\infty)$.

The sets $\mathcal{Y}_c$

- Orb_c^+ is the set of all repelling periodic orbits of f_c.
- For every $\mathcal{O} \in Orb_{c_0}^+$, the function

$$\nu_{\mathcal{O}}(c) := \frac{\rho'_{\mathcal{O}}(c)}{|\mathcal{O}|\,\rho_{\mathcal{O}}(c)} = [\log g_{\mathcal{O}}(c)]'$$

  is defined and analytic around $c = c_0$.
- For each $c \in \mathbb{C}$, we consider the set $\mathcal{Y}_c \subset \mathbb{C}$, defined by

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Theorem (Firsova, G.): The following two properties hold:

(i) For every parameter $c \in \mathbb{C} \setminus \{-2\}$, the set $\mathcal{Y}_c$ is convex; for $c = -2$, the set $\mathcal{Y}_{-2}$ is the union of a convex set and the point $-\frac{1}{6}$.

(ii) For every parameter $c \in \mathbb{C} \setminus \mathbb{M}$, the set $\mathcal{Y}_c$ is bounded. A parameter $c \in \mathbb{C} \setminus \mathbb{M}$ belongs to $\mathcal{X}$, if and only if $0 \in \mathcal{Y}_c$.

Critical points of the Hausdorff dimension function

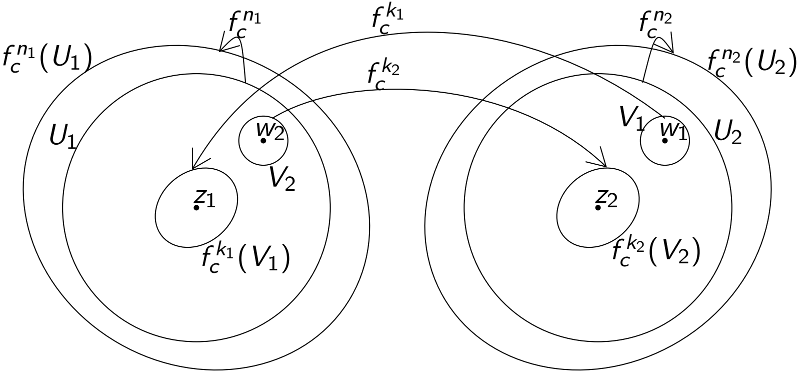
Hausdorff dimension function: $\delta(c) := \dim_H(J_c)$

Theorem (Ruelle): The function δ is real-analytic in each hyperbolic component (including the complement of $\mathbb{M}$).

Theorem (Y. M. He, H. Nie): *(Version for the quadratic family)* If $c \in \mathbb{C}$ is a hyperbolic parameter and $0 \notin \mathcal{Y}_c$, then c is not a critical point of the function δ.

Corollary: The Hausdorff dimension function δ has no critical points in $\mathbb{C} \setminus \mathcal{X}$.

Proof of (i): Averaging Lemma



Averaging Lemma: Let $\mathcal{O}_1, \mathcal{O}_2$ be two distinct non-exceptional repelling periodic orbits of f_c. Then for any $t \in [0,1]$, there exists a sequence of periodic orbits $\mathcal{O}_3, \mathcal{O}_4, \dots$ of f_c, such that

$$g_{\mathcal{O}_j} \to g_{\mathcal{O}_1}^{t} g_{\mathcal{O}_2}^{1-t}, \quad \text{and} \quad \nu_{\mathcal{O}_j} \to t\nu_{\mathcal{O}_1} + (1-t)\nu_{\mathcal{O}_2}$$

uniformly on a neighborhood of c for appropriate branches of the powers.

$\mathbb{M} \subset \mathcal{X}$

- $F_k(c) := f_c^{\circ(k-1)}(c) = f_c^{\circ k}(0)$.

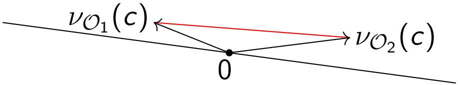
Then $F_k(c)$ is the free term of the polynomial $f_c^{\circ k}(z)$, hence

$$F_k(c) = 2^{-2^k} \prod_{m\in\mathbb{N}, m|k} \prod_{\mathcal{O}\in Orb_c^m} \rho_{\mathcal{O}}(c),$$

where the product is taken over all $m \in \mathbb{N}$, such that m divides k and over all periodic orbits $\mathcal{O} \in Orb_c^m$.

$$\frac{F_k'(c)}{kF_k(c)} = \sum_{m\in\mathbb{N}, m|k} \sum_{\mathcal{O}\in Orb_c^m} \frac{m}{k}\nu_{\mathcal{O}}(c) \to 0, \qquad (2)$$

as $k \to \infty$ over an appropriate subsequence, provided that $c \in \mathrm{int}(\mathbb{M})$ is not parabolic or critically periodic.



Idea: $0 \notin \mathcal{Y}_c$ and Averaging Lemma $\implies$ no convergence in (2).