On Transcendence of Numbers Related to Sturmian and Arnoux-Rauzy Words

P. Kebis, F. Luca, J. Ouaknine, A. Scoones and J. Worrell

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Theorem (Borel 1909)

Almost every number in $[0, 1]$ *is normal.*

Specific Cases

- Champernowne (1933): normal in base-10

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$$\sqrt{2} = 1.41421356237309504880168872420$$

Conjecture (Borel 1950)

Let x be a real irrational algebraic number and $b \geq 2$ a positive integer. Then x is normal in base b.

Conjectures with a Computational Aspect

If the base-b expansion of a real irrational number x is "simple" then x is transcendental.

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Cobham's Second Conjecture (1968)

The base-b expansion of an algebraic number cannot be generated by a morphism of exponential growth.

Tag Machines (Cobham 1968)

- A finite work-tape alphabet,
- B finite output-tape alphabet,
- Start symbol $a \in A$,
- $\sigma : A \to A^*$ morphism, prolongable on a,
- $\varphi : A \to B^*$ letter-to-letter morphism.

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		↓			↓
working	a	b	a	a	c
output	x	y			

Example: The Fibonacci Word

The sequence of finite binary words

$$F_0 = 0, F_1 = 01, F_2 = 010, F_2 = 01001, \ldots$$

satisfying recurrence

$$F_n = F_{n-1} F_{n-2} \quad (n \geq 2)$$

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Converges to infinite Fibonacci word

$$F_\infty = 0100101001001010010100100101 0\ldots$$

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- Fibonacci word is **morphic**: $F_\infty = \lim_{n \to \infty} \sigma^n(0)$, where $\sigma : \{0,1\}^* \to \{0,1\}^*$ is given by $\sigma(0) = 01$ and $\sigma(1) = 0$.

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- Incidence matrix
$$M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
has spectral radius > 1, so σ has **exponential growth**.

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Theorem (Danilov 1972)

Let $\boldsymbol{u}$ be the Fibonacci word. Then for all integers $b \geq 2$ the number

$$S_b(\boldsymbol{u}) := \sum_{n=0}^{\infty} \frac{u_n}{b^n}$$

is transcendental.

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$$x_n := \begin{cases} 1 & \text{if } R_\theta^n(x) \in [0, \theta) \\ 0 & \text{otherwise} \end{cases}$$

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- Sequence is Sturmian of **slope** θ iff it is coding of some x

Arnoux-Rauzy Words

Let $\Sigma = \{0, \ldots, k-1\}$ for some $k \geq 2$. A sequence $\boldsymbol{u} \in \Sigma^\omega$ is **Arnoux-Rauzy** if

- it is uniformly recurrent
- it has subword complexity $p(n) = (k-1)n + 1$
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The **Tribonacci word** is the limit of the infinite sequence defined by recurrence

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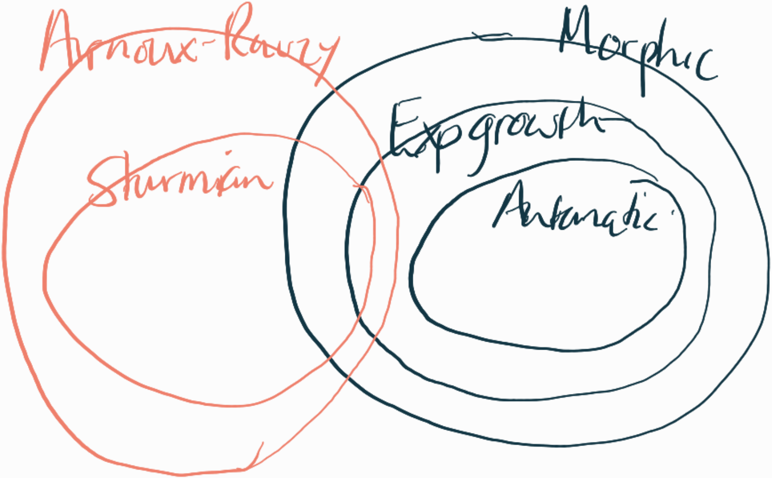
Also generated by the morphism $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$.

Taxonomy of Simple Words



Transcendence of Sturmian Words

Theorem (Ferenczi and Mauduit 1997)

Let $b \geq 2$ be an integer and let $\boldsymbol{u} \in \{0, 1, \ldots, b-1\}^\omega$ be a Sturmian word (more generally, an Arnoux-Rauzy word). Then $S_b(\boldsymbol{u}) := \sum_{n=0}^{\infty} \frac{u_n}{b^n}$ is transcendental.

Diophantine Exponent

Definition (Adamczewski and Bugeaud 2007)

The **Diophantine exponent** of $\boldsymbol{u}$ is the supremum of all real ρ such that $\boldsymbol{u}$ has arbitrarily long prefixes of the form UV^α, for $\alpha \geq 1$, satisfying

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Theorem (Adamczewski-Bugeaud-Luca (reformulated))

For an integer $b \geq 2$ and sequence $\boldsymbol{u} \in \{0, \ldots, b-1\}$, if $\mathrm{Dio}(\boldsymbol{u}) > 1$ then $S_b(\boldsymbol{u})$ is either rational or transcendental.

Approximation by Periodic Words

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- [Adamczewski, Cassaigne, Le Gonidec 2020] shows that words generated by morphims of exponential growth have Diophantine exponent > 1.

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Proposition

If α is rational then there exists $C > 0$ that every rational number a/b different from α satisfies $|\alpha - \frac{a}{b}| > \frac{C}{b}$.

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Theorem (Thue-Siegel-Roth)

Let α be irrational algebraic and $\varepsilon > 0$. There exists $C > 0$ such that $|\alpha - \frac{a}{b}| > \frac{C}{b^{2+\varepsilon}}$ for all a, b.

Diophantine Approximation

Theorem (Schlickewei 75)

Let $m \geq 2$ be an integer, ε a positive real, and S a finite set of prime numbers. Let $L_1, \ldots, L_m$ be linearly independent linear forms with real algebraic coefficients. Then the set of solutions $\boldsymbol{x} \in \mathbb{Z}^m$ of the inequality

$$\left(\prod_{i=1}^{m}\prod_{p\in S}|x_i|_p\right)\cdot\prod_{i=1}^{m}|L_i(\boldsymbol{x})| \leq (\max\{|x_1|,\ldots,|x_m|\})^{-\varepsilon}$$

are contained in finitely many proper linear subspaces of $\mathbb{Q}^m$.

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5. Apply Subspace Theorem to conclude that α is **rational**

Transcendence Results over an Algebraic Base

A. Rényi. Representations for real numbers and their ergodic properties. *Acta. Math. Acad. Sci. Hungar.* **8** (1957).

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Theorem (Adamczewski and Bugeaud 2007a)

Let β be a Pisot or a Salem number and let $\mathrm{Dio}(\boldsymbol{u}) > 1$. Then $S_\beta(\boldsymbol{u})$ either lies in $\mathbb{Q}(\beta)$ or is transcendental.

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Theorem (Adamczewski and Bugeaud 2007b)

Let β be an algebraic integer with $|\beta| > 1$. If $\mathrm{Dio}(\boldsymbol{u}) > \frac{\log M(\beta)}{\log |\beta|}$. Then $S_\beta(\boldsymbol{u})$ either lies in $\mathbb{Q}(\beta)$ or is transcendental.

Our Main Results

Theorem

Let β be algebraic with $|\beta| > 1$. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ be Sturmian sequences, all having the same slope and such that no sequence is a tail of another. Then $\{1, S_\beta(\boldsymbol{u}_1), \ldots, S_\beta(\boldsymbol{u}_k)\}$ is linearly independent over $\overline{\mathbb{Q}}$.

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Corollary

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Theorem

Let $\boldsymbol{u}$ be the d-bonacci sequence. Then for any algebraic number β with $|\beta| > 1$ the sum $S_\beta(\boldsymbol{u}) = \sum_{n=0}^{\infty} \frac{u_n}{\beta^n}$ is transcendental.

Diophantine Approximation Modulo Errors

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- Mismatches come in consecutive symmetric pairs
- Gaps between these pairs expand with n

Tribonacci Word

$$T_\infty := 01020100102010\textcolor{red}{\mathbf{102}}0\textcolor{red}{\mathbf{10}}0\textcolor{red}{\mathbf{102}}01020100102010\textcolor{red}{\mathbf{102010}}0102\ldots$$

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As before, there is a finite alphabet of "mismatches":

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \quad \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 2\\2 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}2 & 0 & 1 & 0\\0 & 1 & 0 & 2\end{bmatrix}$$

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Expanding gaps between *groups of mismatches*

Echoing Sequences

Definition

A sequence $\boldsymbol{u}$ is **echoing** if for all $\rho > 0$ and $\varepsilon > 0$ there exist $d > 0$ and sequences $\langle r_n \rangle_{n=0}^{\infty}$ and $\langle s_n \rangle_{n=0}^{\infty}$ of positive integers and $d \geq 2$ such that:

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- E1 $\langle r_n \rangle_{n=0}^{\infty}$ is unbounded and $s_n \geq \rho r_n$ for all n;
- E2 the set of mismatches between strings $u_0 \dots u_{s_n}$ and $u_{r_n} \dots u_{r_n + s_n}$ is a contained in a union of at most d intervals of total length at most εs_n.

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- E2 the set of mismatches between strings $u_0 \ldots u_{s_n}$ and $u_{r_n} \ldots u_{r_n + s_n}$ is a contained in a union of at most d intervals of total length at most εs_n.
- E3 the gaps between intervals expand with n.

Echoing Sequences

Definition

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Use Subspace Theorem to show transcendence of $S_\beta(\boldsymbol{u})$ for $\boldsymbol{u}$ echoing.

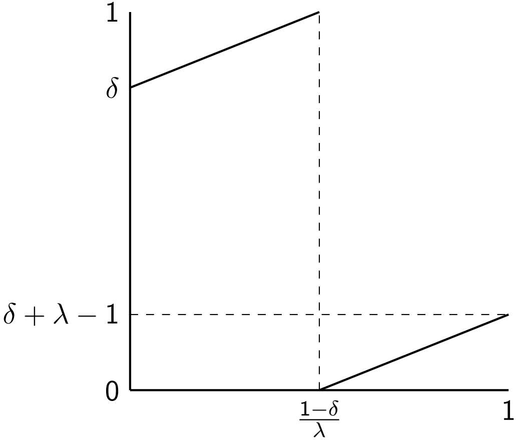
Application to Dynamical Systems

"Are all irrational elements of the Cantor ternary set transcendental?"

K. Mahler, Some suggestions for further research, *Bull. Austral. Math. Soc.* 29 (1984).

Contracted Rotations

Given $0 < \lambda, \delta < 1$ such that $\lambda + \delta > 1$, map $f : I \to I$ given by $f(x) := \{\lambda x + \delta\}$ is a **contracted rotation** with **slope** λ and **offset** δ.



Cantor Sets from Rotations

Rotation Number

Consider the limit set $C := \bigcap_{n=0}^{\infty} f^n(I)$. Then f has a **rotation number** θ such that restriction of f to C is conjugate to the rotation map R_θ and $\overline{C}$ is a Cantor set if θ is irrational.

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Theorem (Luca, Ouaknine, W., 2023)

If f has algebraic slope and irrational rotation number then every element of the Cantor set $\overline{C}$ other than 0 and 1 is transcendental.

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- Generalises result of Bugeaud, Kim, Laurent, Nogueira, which had $\lambda^{-1} \in \mathbb{Z}$.

Application to LTI Reachability

Consider LTI system in $\mathbb{R}^2$ with

- Control polyhedron: $U := [0,1] \times \{0\}$
- Transition matrix $A := \frac{1}{b}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

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Determine whether $\sum_{n=0}^{\infty} u_n \frac{\cos(n\theta)}{b^n} \geq c$, where $u_n = 1$ if $\cos(n\theta) \geq 0$ and $u_n = 0$ otherwise.