Explosive Condensation in a One-dimensional Particle System

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Plan:

- **Real Space Condensation**
  - Zero Range Process
  - Factorised Steady State (**FSS**)
  - Condensation and large deviation of sums of random variables

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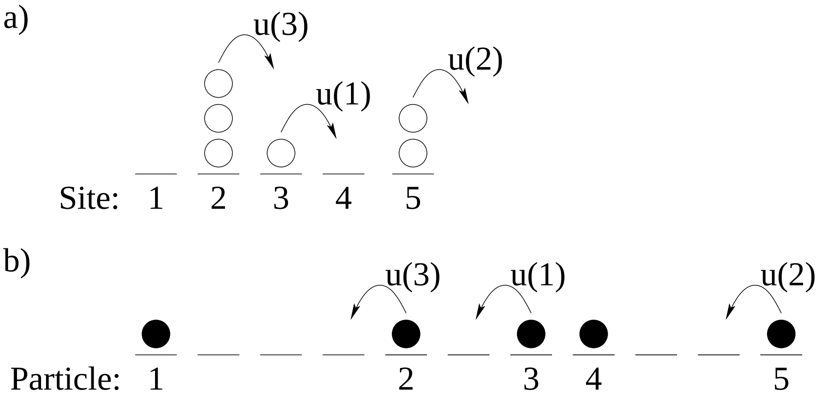
References:
T Hanney and M.R. Evans, J. Phys. A 2005
M. R. Evans, S. N. Majumdar and R. K. P. Zia J. Stat. Phys. 2006
B. Waclaw and M. R. Evans, Phys. Rev. Lett. 2012

Zero-Range Process



a) "balls-in-boxes" picture

b) "Exclusion Process" picture

Motivation for ZRP

- Specific physical systems map onto ZRP

  e.g. polymer dynamics, sandpile dynamics, traffic flow

- Effective description of dynamics involving exchange between domains

  e.g. phase separation dynamics

- Factorised Steady State (system of L sites and N particles)

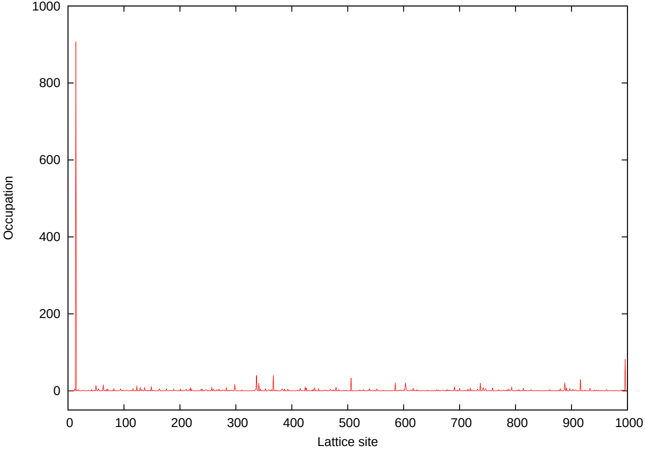
$$\mathrm{P}(m_1.....m_L) = \frac{1}{Z_{N,L}} f(m_1)\dots f(m_L)\,\delta(\sum_i m_i - N)$$

where the single-site weight $f(m)$

$$f(m) = \prod_{n=1}^{m} \frac{1}{u(n)}$$

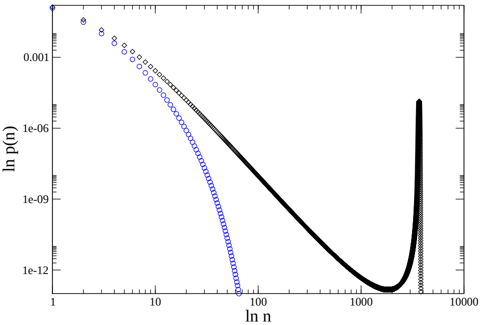
Real Space Condensation

Snapshot of ZRP $u(m) = 1 + \frac{3}{m}$



Real Space Condensation

Single-site mass distribution in ZRP $u(m) = 1 + \frac{5}{m}$



below critical density

above critical density

Real Space Condensation

Grand Canonical Ensemble: $p(m) = Az^m f(m)$ $z < 1$ z is fugacity

Constraint: $$\sum_{m=0}^{\infty} mp(m) = \rho \equiv \lim_{L,N\to\infty} \frac{N}{L}$$

i.e. density $\rho(z)$ as function of z

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Then $z \to 1$ gives the max allowed value of density $\rho_{\max}$

$$\rho_{\max} \to \infty \qquad \text{if} \quad \gamma \leq 2$$

$$\rho_{\max} \to \rho_c \quad < \infty \qquad \text{if} \quad \gamma > 2$$

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Thus for $\gamma > 2$ we have condensation if $\rho > \rho_c$

Nature of the Condensate: a large deviation effect

Canonical partition function: (computed in EMZ 2006)

$$Z_{N,L} = \sum_{\{m_i=0\}}^{\infty} \prod_{i}^{L} f(m_i) \delta\left(\sum_{j}^{L} m_j - N \right)$$

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Condensate shows up in a large deviation of a sum of random variables

when $N \gg \mu_1 L$ with $\sum_{m=0}^{\infty} m\, f(m) \equiv \mu_1 < \infty$

Results for condensate bump scaling laws

$3 > \gamma > 2$

$$p_{\rm cond} \simeq \frac{1}{L}\frac{1}{L^{1/(\gamma-1)}} V_\gamma(z) \qquad z = \frac{(m - M_{ex})}{L^{1/(\gamma-1)}}$$

$$V_\gamma = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \exp(-zs + A\Gamma(1-\gamma)s^{\gamma-1})$$

strongly asymmetric

$\gamma > 3$

$$p_{\rm cond} \simeq \frac{1}{L}\frac{1}{\sqrt{2\pi\Delta^2 L}} \exp(-\frac{z^2}{2\Delta^2}) \qquad z = \frac{(m - M_{ex})}{L^{1/2}}$$

gaussian

N.B. in all cases $\displaystyle\int p_{\rm cond}(m)\,{\rm d}m = \frac{1}{L}$.

For rigorous work see also Grosskinsky, Schutz, Spohn JSP 2003, Ferrari, Landim, Sisko JSP 2007, Armendariz and Loukakis PTRF 2009, Beltran and Landim 2011

Physical Systems with Real-space Condensation:

- Traffic and Granular flow (O'Loan, Evans, Cates, 1998)

- Cluster Aggregation and Fragmentation (Majumdar et al 1998)

- Granular clustering (van der Meer et al, 2000)

- Phase separation in driven systems (Kafri et al, 2002).

- Socio-economic contexts: company formation, city formation, wealth condensation etc. (Burda et al, 2002)

- Networks (Dorogovstev & Mendes, 2003,....)

- ...

II Explosive Condensation

Consider Generalisation of ZRP to dependence on target site.

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We still have factorised stationary state if $u(m,n)$ satisfy :

$$u(m,n) = u(n+1,m-1)\frac{u(1,n)u(m,0)}{u(n+1,0)u(1,m-1)}$$

$$u(m,n) - u(n,m) = u(m,0) - u(n,0)$$

and the single-site weight becomes

$$f(m) = Az^m \prod_{k=1}^{m} \frac{u(1,k-1)}{u(k,0)}$$

Explosive Condensation cont.

A simple form which gives a factorised stationary state is

$$u(m,n) = [v(m) - v(0)]v(n)$$

then the single-site weight becomes

$$f(m) \propto \prod_{k=1}^{m} \frac{v(k-1)}{v(k) - v(0)}$$

For f to decay as $f(m) \sim m^{-\gamma}$ (for condensation) we now have several possible choices of asymptotic behaviour of $v(m)$

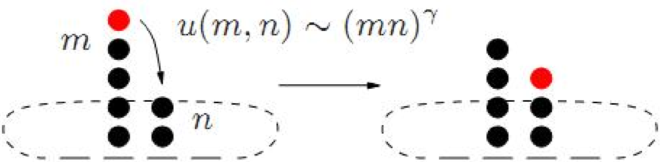
$$v(m) \simeq 1 + \frac{\gamma}{m} \qquad \text{'ZRP like'}$$

$$v(m) \sim m^{\gamma} \qquad \text{'explosive'}$$

Explosive dynamics

$$u(m,n) = [v(m) - v(0)]v(n)$$

with $v(m) = (\epsilon + m)^{\gamma}$ and $\epsilon > 0$

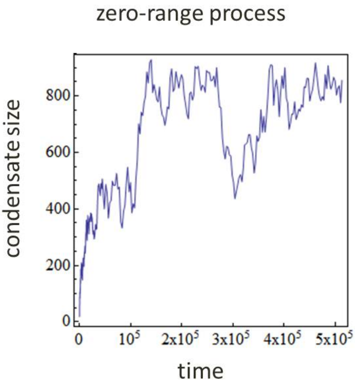


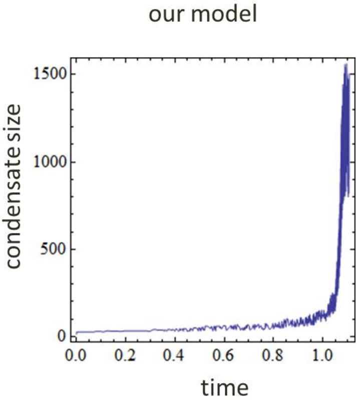
Get condensation for $\gamma > 2$.

c.f. Inclusion process: $\gamma = 1$ and $\epsilon \to 0$

Contrasting Dynamics

Both choices (ZRP-like, explosive) generate same stationary state (condensed) but the dynamics are very different:





$T_{SS} \sim L^2$ $\qquad$ $T_{SS} = ?$

Explosive Dynamics

Speed of condensate $v(m) \sim m^{\gamma}$ 'slinky motion'
c.f. non-Markovian ZRP (Hirschberg, Mukamel, Schutz 2009)

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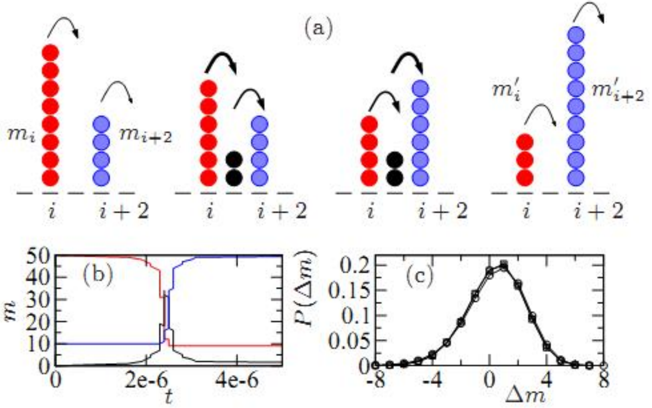
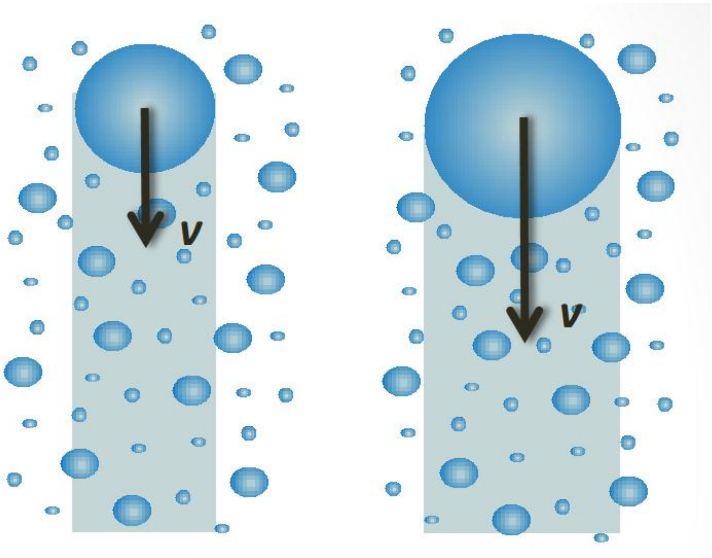
Scattering collisions between two condensates



- Almost elastic scattering
- Larger condensate picks up mass

Raindrops



Heuristic/Approximate Picture

- Initially a large number $O(L)$ of clusters (mini-condensates) emerge from initial condition
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- These grow in time - first out of these to become macroscopic determines relaxation time T
- Relaxation time for a putative condensate comes from simplistic picture of infinite sequence of collisions where condensate accrues mass:

$$m_n = m_{n-1} + \delta \quad \text{deterministic accretion}$$

$$t_n = t_{n-1} + \Delta t_n \quad \text{stochastic accretion times}$$

$$\text{where} \quad p_n(\Delta t_n) = \lambda_n \mathrm{e}^{-\lambda_n \Delta t_n} \quad \text{and} \quad \lambda_n = A m_n^{\gamma} \quad \text{(speed)}$$

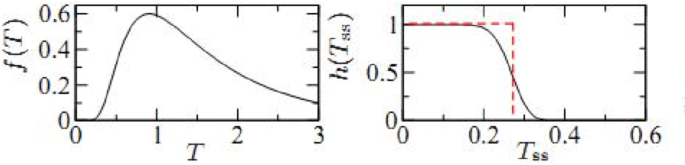
Then distribution of $T = \sum_{n=1}^{\infty} \Delta t_n$ is given by

$$f(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \mathrm{e}^{-i\omega T} \prod_{n=1}^{\infty} \frac{1}{1 - i\omega/\lambda_n}$$

Heuristic/Approximate Picture cont

Using the identity $\prod_{k=1}^{\infty}\left(1-\frac{x}{k^n}\right)^{-1} = -x^n \prod_{k=0}^{n-1}\Gamma(-\mathrm{e}^{2\pi ik/n}x^{1/n})$ the integral may be estimated by saddle point and one finds for small T

$$f(T) \simeq CT^{\frac{(1-3\gamma)}{2(\gamma-1)}} \exp -AT^{-1/(\gamma-1)}$$



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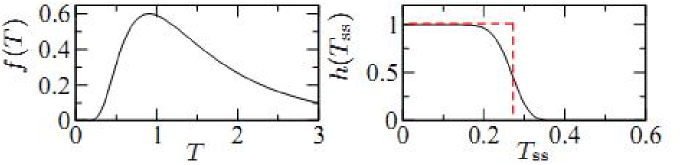

Extreme value statistics for average of *minimum* of L iidrvs drawn from $f(T)$ implies $L\int_0^{T_{\min}} f(T)\,\mathrm{d}T = 1$ which gives

$$T_{\min} \sim (\ln L)^{1-\gamma}$$

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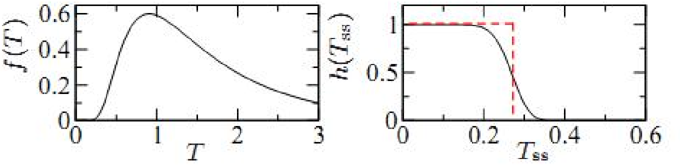

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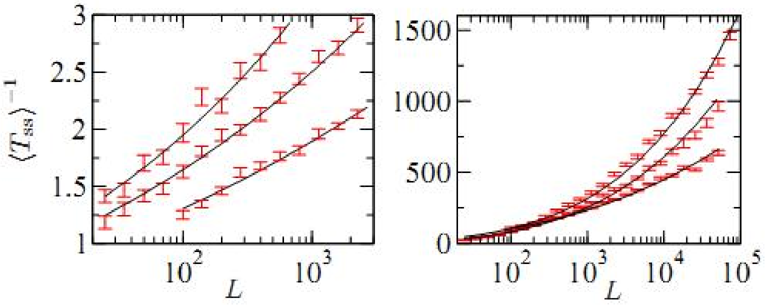
Instantaneous as $L \to \infty$

Numerical Evidence for instantaneous condensation



$\langle Tss \rangle - 1$ obtained in numerical simulations (points) and from formula $c_2(c_3 + \ln L)^{1-\gamma}$ fitted to data points (lines). In all cases the density $\rho =$ 2 and $\gamma = 3$, 4, 5 (curves from bottom to top). Left: $v(m) = (0.3 + m)$, every 5th site has initially 10 particles. Right: $v(m) = (1 + m)$ particles are distributed randomly in the initial state. $\langle Tss \rangle - 1$ for different γ differ by orders of magnitude and hence they have been rescaled to plot

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- Real space condensation — ubiquitous dynamical phase transition in variety of contexts

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- Understanding in terms of large deviations of sum of random variables
- Explosive Condensation has same stationary state as ZRP but relaxation time $T \sim (\ln L)^{1-\gamma}$ vanishes for large L
- First (?) spatially extended realisation of the instantaneous gelation phenomenon seen in mean-field models of cluster aggregation (Smoluchowski equation)

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = \frac{1}{2}\sum_{j+k=i} K_{jk} N_j N_k - \sum_j K_{ij} N_i N_j$$

where e.g. $K_{ij} = i^\nu j^\mu + i^\mu j^\nu$