TMUA 2021 Paper 1 Question 11

The function f is given by

$$f(x) = x^{\frac{1}{7}}(x^2 - x + 1)$$

Find the fraction of the interval $0 < x < 1$ for which $f(x)$ is decreasing.

A $\frac{2}{15}$

B $\frac{1}{5}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

F $\frac{4}{5}$

G $\frac{13}{15}$

When a function is decreasing, it has a negative gradient. Therefore, we start by considering the question: "for what values of x (between 0 and 1), does $f(x)$ have a negative gradient?"

$$f(x) = x^{\frac{1}{7}}(x^2 - x + 1)$$
$$= x^{\frac{15}{7}} - x^{\frac{8}{7}} + x^{\frac{1}{7}}$$

Now we find the derivative function, $f'(x)$

$$f'(x) = \frac{15}{7}x^{\frac{8}{7}} - \frac{8}{7}x^{\frac{1}{7}} + \frac{1}{7}x^{\frac{-6}{7}}$$

$$= \frac{1}{7}x^{\frac{1}{7}}(15x - 8 + x^{-1})$$ taking out a factor of $\frac{1}{7}x^{\frac{1}{7}}$

$$= \frac{1}{7}x^{\frac{-6}{7}}(15x^2 - 8x + 1)$$ taking out a further factor of $\frac{1}{x} = x^{-1}$

$$= \frac{1}{7}x^{\frac{-6}{7}}(5x-1)(3x-1)$$

this factor is always positive for $0 < x < 1$

$5x-1=0$ when $x = \frac{1}{5}$

$3x-1=0$ when $x = \frac{1}{3}$

x	$0 < x < \frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5} < x < \frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3} < x < 1$
$\frac{1}{7}x^{\frac{-6}{7}}$	+	+	+	+	+
$5x-1$	−	0	+	+	+
$3x-1$	−	−	−	0	+
$f'(x)$	+	0	−	0	+

FIGURE 1

FIGURE 1 shows the sign of $f'(x)$ for intervals within $0 < x < 1$. This is called a "sign table" and is used to determine values of the domain, for which a function is positive or negative.

From FIGURE 1 we see that $f'(x)$ is negative, when $\frac{1}{5} < x < \frac{1}{3}$ i.e. these are the values of x for which $f(x)$ is decreasing

This represents the following proportion of the interval $0 < x < 1$

$$\frac{\frac{1}{3} - \frac{1}{5}}{1} = \frac{\frac{5}{15} - \frac{3}{15}}{1} = \frac{2}{15}$$

So the correct answer is A