

TMUA 2021 Paper 1 Question 20

Find the length of the curve with equation

$$2 \log_{10}(x - y) = \log_{10}(2 - 2x) + \log_{10}(y + 5)$$

- A 5
 - B 10
 - C 15
 - D 3π
 - E 9π
 - F 12π
- Log functions are only valid for positive inputs, so we must have $x - y > 0$, $2 - 2x > 0$ and $y + 5 > 0$ i.e. we must have $x > y$, $x < 1$ and $y > -5$

Now, we simplify the given equation using the laws of logs as follows:

$$\log_{10}(x - y)^2 = \log_{10}((2 - 2x)(y + 5))$$

Annotations:
 - A cloud around $\log_b a = \log_b a^n$ points to the left side.
 - A cloud around $\log_b a + \log_b c = \log_b ac$ points to the right side.

raising both sides of this equation to a base of 10, gives

$$(x - y)^2 = (2 - 2x)(y + 5)$$

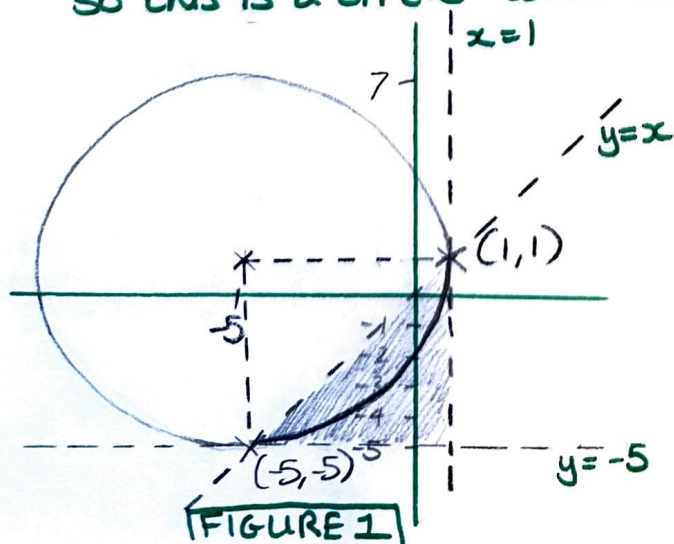
$$\Leftrightarrow x^2 - 2xy + y^2 = 2y - 2xy + 10 - 10x$$

$$x^2 + 10x + y^2 - 2y - 10 = 0$$

these terms indicate that this is the equation of a circle. Completing the square on the x terms and y terms should make the radius and circle centre clear.

$$(x + 5)^2 + (y - 1)^2 - 36 = 0$$

so this is a circle with centre at $(-5, 1)$ and radius 6



Keeping in mind our constraints $x > y$, $x < 1$ and $y > -5$

Figure 1 shows that the length we are asked to find is the arc that lies in the shaded region, that is $\frac{1}{4}$ the circumference.
 $\frac{1}{4} \times 2\pi r = \frac{1}{2} \pi (6) = 3\pi$

so the correct answer is D