TMUA 2021 Paper 2 Question 16

p and q are real numbers, and the equation

$$x|x| = px + q$$

has exactly k distinct real solutions for x.

Which one of the following is the complete list of possible values for k?

A 0, 1, 2

B 0, 1, 2, 3

C 0, 1, 2, 3, 4

D 0, 2, 4

E 1, 2, 3

F 1, 2, 3, 4

We will approach this question graphically.
The right hand side of the equation is linear (consider how it has the same structure as $y=mx+c$)
The left hand side of the equation involves $|x|$ and we can graph this by considering the two separate cases

① when $x \geq 0$, $x|x| = x^2$

② when $x < 0$, $x|x| = -x^2$

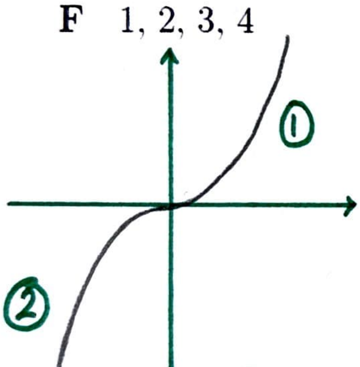

FIGURE 1

FIGURE 1 shows the graph.
Now we consider the possibilities for how many times a line could intersect the graph in FIGURE 1

The graph in FIGURE 1 extends infinitely both in the positive y direction and in the negative y direction, therefore guaranteeing at least one intersection with any line.
This rules out options A, B, C and D

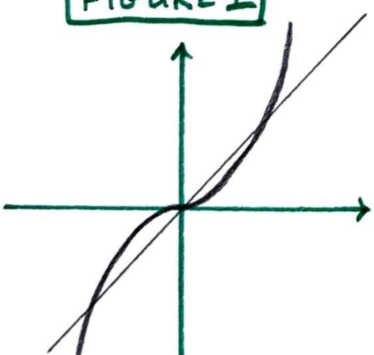
FIGURE 2

Now we consider whether there exists a line which would give 4 intersections, to decide which out of options E and F is correct.

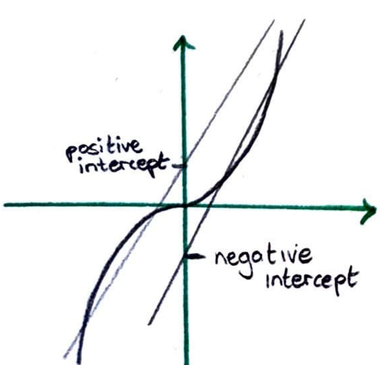

FIGURE 3

First we note that if such a line exists, it must have positive gradient.
Any line through the origin with positive gradient would give exactly 3 intersections (this is due to the symmetry of the graph of $x|x|$, see FIGURE 2 for an example of this).

Now we note that in order to achieve 4 intersections, we would need 2 from the part of the graph for which $x>0$ and 2 from the part of the graph for which $x<0$

Any line that intersects the graph twice when $x>0$, must have a negative y-intercept
Any line that intersects the graph twice where $x<0$ must have a positive y-intercept

see figure 3 for an example of this

But these are mutually exclusive outcomes
Therefore it is not possible to achieve 4 intersections

So the correct answer is E.