TMUA 2021 Paper 2 Question 17

Consider the following functions defined for $x > 1$:

$$f(x) = \log_2(\log_2 \sqrt{x})$$

$$g(x) = \log_2(\sqrt{\log_2 x})$$

Which one of the following is true for **all** values of $x > 1$?

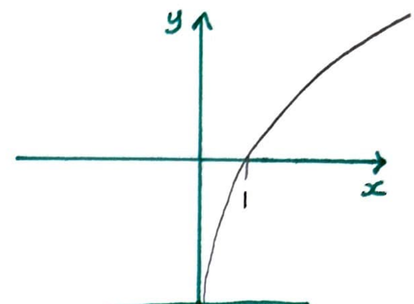

FIGURE 1

A $0 \le f(x) \le g(x)$ **or** $g(x) \le f(x) \le 0$

B $0 \le g(x) \le f(x)$ **or** $f(x) \le g(x) \le 0$

C $\frac{1}{2} \le f(x) \le g(x)$ **or** $g(x) \le f(x) \le \frac{1}{2}$

D $\frac{1}{2} \le g(x) \le f(x)$ **or** $f(x) \le g(x) \le \frac{1}{2}$

E $1 \le f(x) \le g(x)$ **or** $g(x) \le f(x) \le 1$

F $1 \le g(x) \le f(x)$ **or** $f(x) \le g(x) \le 1$

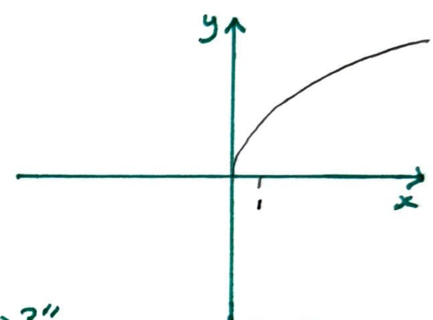

FIGURE 2

Let's start by considering the question "are there any values of x for which $f(x)=g(x)$?"

We have

$$f(x) = g(x)$$

$$\iff \log_2(\log_2\sqrt{x}) = \log_2(\sqrt{\log_2 x})$$

we can think of this step as raising both sides of the equation to base 2

$$\iff \log_2\sqrt{x} = \sqrt{\log_2 x}$$

using logarithm laws and $\log_2\sqrt{x} = \log_2(x)^{\frac{1}{2}}$

$$\iff \frac{1}{2}\log_2 x = \sqrt{\log_2 x}$$

we can square both sides because $x>1$ which means both sides of the equation on the previous line are positive

$$\iff \frac{1}{4}(\log_2 x)^2 = \log_2 x$$

remember: $\log_2 x \neq 0$ because $x>1$

$$\iff \frac{1}{4}\log_2 x = 1$$

$$\iff \log_2 x = 4$$

$$\iff x = 16$$

Now we know that when $x=16$, $f(x)=g(x)$

$f(16) = \log_2(\log_2\sqrt{16}) = \log_2(\log_2 4) = \log_2 2 = 1$

Therefore $g(16) = 1$

Since we know that $f(16)=g(16)=1$, we can narrow our choices down to options E or F.

Now we can investigate whether we have $f(x) < g(x)$ or $g(x) < f(x)$, when both functions are greater than 1, that is when $x > 16$

Choosing a value of x which is a power of 2 will make the calculations easier

Let's choose $x=64$

$$f(64) = \log_2(\log_2\sqrt{64}) = \log_2(\log_2 8) = \log_2 3$$
$$g(64) = \log_2(\sqrt{\log_2 64}) = \log_2\sqrt{6}$$

since $\sqrt{6} < 3$, $\log_2\sqrt{6} < \log_2 3$

Therefore when $x=64$, we have $g(64) < f(64)$

Now $f(x)$ and $g(x)$ are both increasing functions and with this in mind we can say that

consider that $f(x)$ and $g(x)$ are both composite functions involving $\log_2 x$ and $\sqrt{x}$, see FIGURE 1 for a graph of $\log_2 x$ and FIGURE 2 for a graph of $\sqrt{x}$, which are both increasing functions

$$1 \leq g(x) \leq f(x)$$

so the correct answer is F