TMUA 2021 Paper 2 Q19

The angle θ can take any of the values $1°, 2°, 3°, \ldots, 359°, 360°$.

For how many of these values of θ is it true that

$$\sin\theta\sqrt{1+\sin\theta}\sqrt{1-\sin\theta}+\cos\theta\sqrt{1+\cos\theta}\sqrt{1-\cos\theta}=0$$

A 0

B 1

C 2

D 4

E 93

F 182

G 271

H 360

$\sin\theta\sqrt{(1+\sin\theta)(1-\sin\theta)}+\cos\theta\sqrt{(1+\cos\theta)(1-\cos\theta)}=0$

$\sin\theta\sqrt{1-\sin^2\theta}+\cos\theta\sqrt{1-\cos^2\theta}=0$

difference of two squares
ie. $1-\sin^2\theta=(1+\sin\theta)(1-\sin\theta)$
and similarly for $\cos\theta$

$\sin\theta\sqrt{\cos^2\theta}+\cos\theta\sqrt{\sin^2\theta}=0$

$\sin^2\theta+\cos^2\theta=0$

Now $\sqrt{\cos^2\theta}$ is equivalent to $|\cos\theta|$
and $\sqrt{\sin^2\theta}$ is equivalent to $|\sin\theta|$
we can see this by considering that
$\sqrt{\cos^2\theta}\geq 0$, regardless of whether $\cos\theta\geqslant 0$
or $\cos\theta<0$

case	$\sin\theta$	$\cos\theta$
1	$\geqslant 0$	$\geqslant 0$
2	$\geqslant 0$	<0
3	<0	$\geqslant 0$
4	<0	<0

TABLE 1

We need to consider the 4 cases listed in TABLE 1

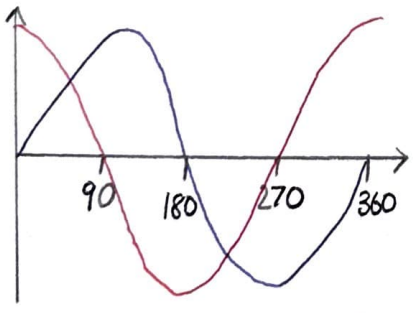
Case 1 gives $\sin\theta\cos\theta+\cos\theta\sin\theta=0$

$\Leftrightarrow 2\sin\theta\cos\theta=0$

$\Leftrightarrow \sin\theta\cos\theta=0$

and this holds when $\sin\theta=0$ or $\cos\theta=0$
that is for $\theta=90, 180, 270, 360$

note from the question that $\theta\neq 0$



$y=\cos\theta$ (in red)

$y=\sin\theta$ (in purple)

FIGURE 1

Case 4 also results in $\sin\theta\cos\theta=0$
so no further values of θ would be found for case 4.

Case 2 gives $-\sin\theta\cos\theta+\cos\theta\sin\theta=0$

and

Case 3 gives $\sin\theta\cos\theta-\cos\theta\sin\theta=0$

which both result in $\sin\theta\cos\theta-\sin\theta\cos\theta=0$

Therefore all values of θ for which cases 2 and 3 hold will satisfy the original equation. i.e. values of θ for which $\sin\theta \geq 0$ and $\cos\theta < 0$ (case 3) and values of θ for which $\sin\theta > 0$ and $\cos\theta \geq 0$ (case 4).

From FIGURE 1 we can see that

$90 \leq \theta \leq 180$ for case 2 (exactly 91 values)

and

$270 \leq \theta \leq 360$ for case 3 (exactly 91 values)

note: this includes the values we found earlier

note also: θ can only take integer values

This means the total number of values of θ for which the given equation is true is $91 + 91 = 182$

Therefore the correct answer is option F.