TMUA 2021 Paper 2 Q20

A sequence of functions $f_1, f_2, f_3, \ldots$ is defined by

$$f_1(x) = |x|$$

$$f_{n+1}(x) = |f_n(x) + x| \quad \text{for } n \geq 1$$

Find the value of

$$\int_{-1}^{1} f_{99}(x)\,dx$$

A 0

B 0.5

C 1

D 49.5

E 50

F 99

G 99.5

H 100

Figures 1, 2, and 3 show the graphs of the functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ respectively.

We can see a pattern emerging.

For $x \geq 0$, as n increases, $f_n(x)$ behaves like $y = nx$ for each n.

For $x < 0$, when n is odd $f_n(x)$ behaves like $y = -x$ and when n is even, $f_n(x)$ behaves like $y = 0$.

See FIGURE 4 for a sketch of the graph of $f_{99}(x)$, which behaves like $y = 99x$ for $x \geq 0$ and like $y = -x$ for $x < 0$

The value of the integral for which we are asked, will have the same value as the shaded region in FIGURE 4, that is

$$\frac{1}{2} \times 1 \times 1 + \frac{1}{2} \times 1 \times 99$$

$$= \frac{1}{2} + \frac{99}{2}$$

$$= 50$$

Therefore the correct answer is option E.

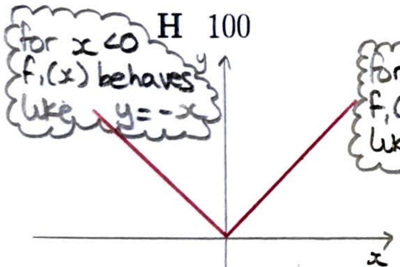

FIGURE 1 $f_1(x)$

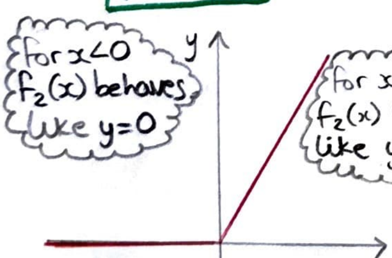

FIGURE 2 $f_2(x)$

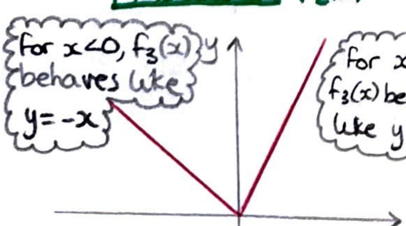

FIGURE 3 $f_3(x)$

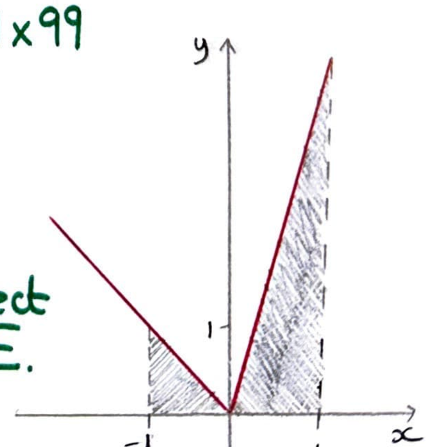

FIGURE 4 $f_{99}(x)$