TMUA 2021 Paper 2 Q9

Consider the following statements about a polynomial $f(x)$:

I $f(x) = px^3 + qx^2 + rx + s$, where $p \neq 0$.

II There is a real number t for which $f'(t) = 0$.

III There are real numbers u and v for which $f(u)\,f(v) < 0$.

Which of these statements is/are **sufficient** for the equation $f(x) = 0$ to have a real solution?

	Statement I is sufficient	Statement II is sufficient	Statement III is sufficient
A	Yes	Yes	Yes
B	Yes	Yes	No
C	Yes	No	Yes
D	Yes	No	No
E	No	Yes	Yes
F	No	Yes	No
G	No	No	Yes
H	No	No	No

Let P be the following statement

P: $f(x)=0$ has a real solution

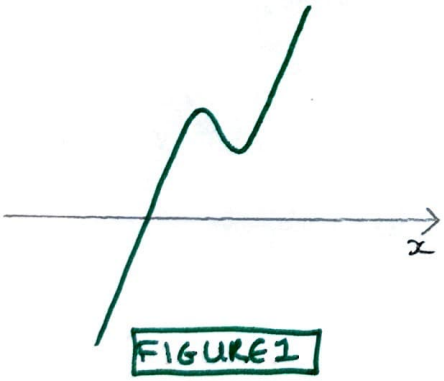

FIGURE 1

For statement I, $p \neq 0$ tells us that $f(x)$ is a cubic. Consider the cubic in figure 1; we can imagine that, no matter where the x-axis lies, the cubic will pass through it at least once. This is true for all cubic polynomials, therefore I is sufficient for P, i.e. $I \Rightarrow P$

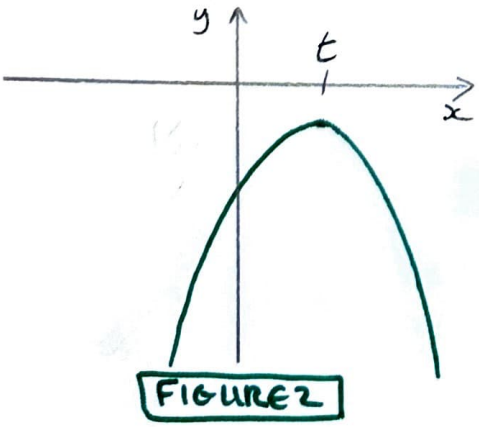

FIGURE 2

For statement II, consider the quadratic in FIGURE 2, which has a stationary point at $(t, f(t))$ but it does not meet the x-axis for any value of x so it satisfies II but does not satisfy P, therefore II is not sufficient for P i.e. $II \not\Rightarrow P$

When we see $f(u)f(v)<0$, it's telling us that $f(u)$ and $f(v)$ have opposite signs i.e. if $f(u)$ is positive, then $f(v)$ must be negative, or if $f(v)$ is positive, then $f(u)$ must be negative. This means there is at least one value of $f(x)$ that is positive and at least one value of $f(x)$ that is negative, therefore $f(x)$ must cross the x-axis at least once, which guarantees P.
This means statement III is sufficient for P, i.e. III $\Rightarrow$ P

In conclusion, we need Yes, No, Yes which means the correct answer is option C.