Lecture 10

Irrotational Flow past a Circular Cylinder

10.1 A Special Complex Potential

Let us analyse the flow resulting from a complex potential of the form

$$\chi = U\left(z + \frac{a^2}{z}\right). \tag{10.1}$$

Note 1: As $z \to \infty$, $\chi \to Uz \Rightarrow u - iv = \partial_z \chi = U \Rightarrow u = U, v = 0$. So, at ∞ we have a uniform flow along the $x - axis$.

Note 2: On the circle $z\bar{z} = a^2$ we have

$$z = \frac{a^2}{\bar{z}}.$$

Substituting this into the second term on the RHS of (10.1) we find

$$\begin{aligned}\chi &= U\left(z + \frac{a^2}{a^2/\bar{z}}\right)\\ &= U(z + \bar{z})\\ &= 2Ux = \phi + i\psi.\end{aligned}$$

This implies that $\psi = 0$ on $|z| = a$. But $\psi = constant$ on the streamlines, i.e. $|z| = a$ *is a streamline.*

In other words, the flow satisfies the boundary condition $u_n = 0$ on the circle $|z| = a$. So the flow with χ given by (10.1) is a flow (uniform at ∞) past a circular cylinder of radius a, figure 10.1.

Note: That $u_t \neq 0$, i.e. there is a finite slip on the boundary.

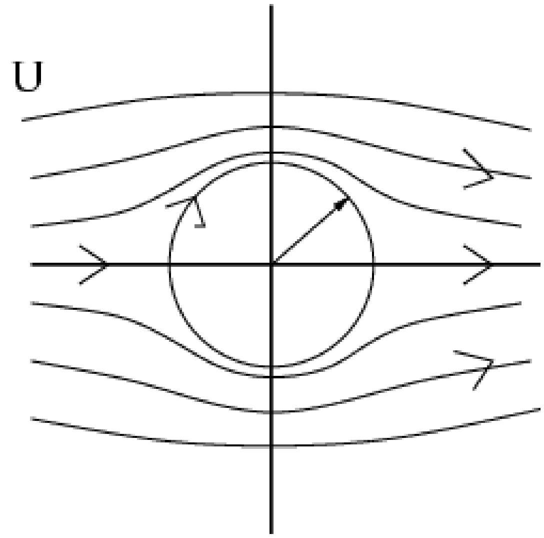


Figure 10.1: Flow Past a Circular Cylinder

10.2 This Solution is not Unique. Why?

This is because, we can add any irrotational axially symmetric flow. Any flow like this is equivalent to a flow generated by a point vortex located at the origin and having some circulation Γ. So,

$$\chi = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z, \tag{10.2}$$

is another solution with arbitrary constant Γ.

10.3 Finding the Velocity Fields

Substituting $z = r\exp(i\theta)$ into expression (10.1) and differentiating, we find

$$\begin{aligned} u_r &= \frac{1}{r}\partial_\theta \psi \\ &= \frac{1}{r}\partial_\theta \left[U\left(r - \frac{a^2}{r}\sin\theta\right)\right] \\ &= U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \end{aligned}$$

and

$$\begin{aligned} u_\theta &= -\partial_r \psi \\ &= -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta. \end{aligned}$$

In a similar manner the complex potential (10.2) gives,

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta,$$

and

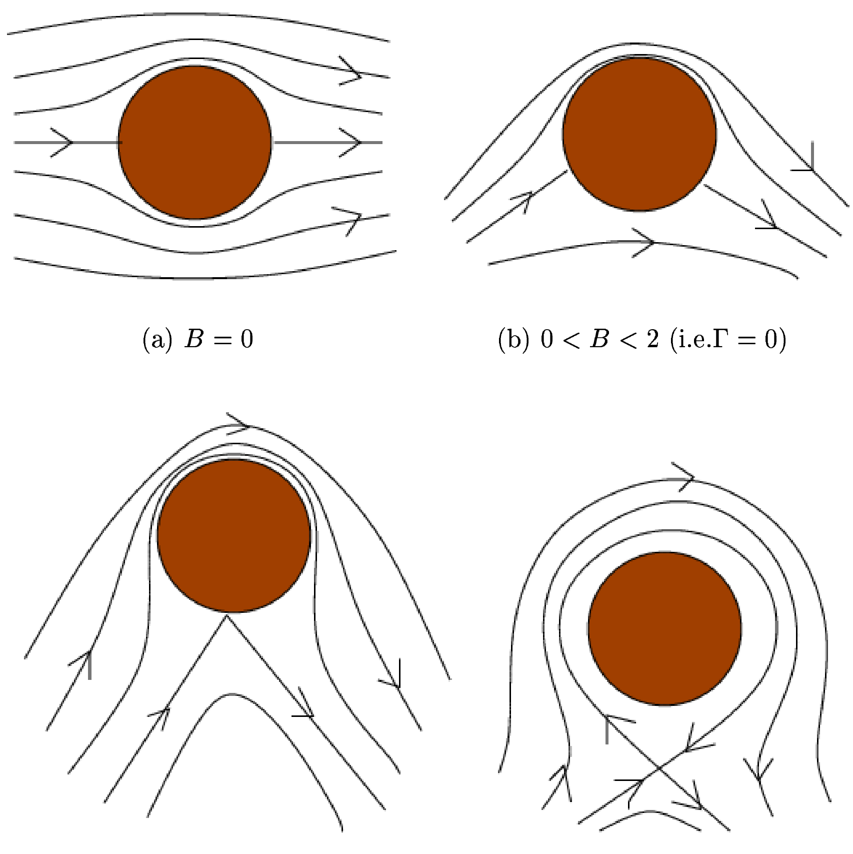
$$u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r},$$

(you should check this).

Note: That $u_r = 0$ at $r = a$, as it has to be.

10.4 Flow Streamlines

The streamlines are given by the lines $\psi = constant$. Let us draw the streamlines for different values of parameter $B = -\Gamma/2\pi U a$, figure 10.2



(a) $B = 0$

(b) $0 < B < 2$ (i.e.$\Gamma = 0$)

(c) $B = 2$

(d) $B > 2$

Figure 10.2: Streamlines

Note: The closed streamlines, case (d), imply that some fluid is trapped near the cylinder.

10.4.1 What Occurs in Nature?

If there is no rotation, the flow should be symmetric $\to \Gamma = 0$, case (a). Rotation breaks the symmetry $\to$ finite Γ. The stronger the rotation the greater Γ.

10.5 Net Force on the Cylinder

First, let us find the pressure, at the surface of the cylinder, using Bernoulli's theorem,

$$p + \frac{1}{2}\rho u^2 = const = C,$$

where C is the same for all streamlines since the fluid is irrotational. On the circle $r = a$ we find,

$$\frac{p}{\rho} = C - 2U^2 \sin^2\theta + \frac{U\Gamma}{\pi a}\sin\theta.$$

Note: That $p(\theta) = p(\pi - \theta)$, so the net force $\mathbf{F}$ is perpendicular to $\mathbf{e}_x$ (i.e. $F_x = 0$). This is a manifestation of a more general fact known as the *D'Alembert paradox*. That is, there is no drag related with any irrotational inviscid flow.

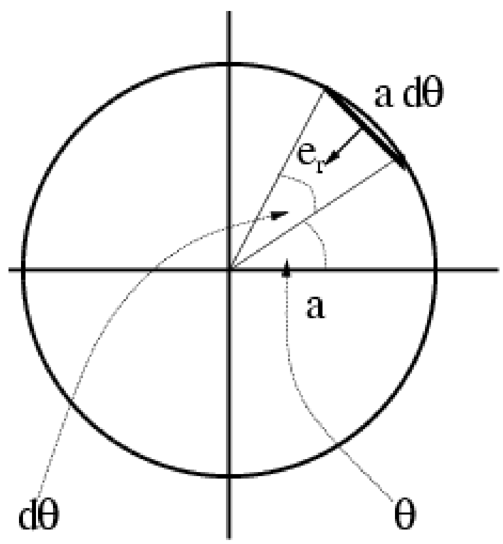


Figure 10.3:

Consider figure 10.3, The force $\delta\mathbf{F}$ on the surface element $ad\theta$ (unit length on z) is,

$$\delta\mathbf{F} = pad\theta\mathbf{e}_r.$$

The y component of this is,

$$\delta F_y = -pa\sin\theta d\theta.$$

Therefore, the net force is,

$$\begin{aligned} F &= -\int_0^{2\pi} pa\sin\theta d\theta \\ &= \rho\int_0^{2\pi}\left(2U^2\sin^2\theta - \frac{U\Gamma}{\pi a}\sin\theta\right) a\sin\theta d\theta \\ &= -\rho U\Gamma. \end{aligned}$$

Note: That $F = O$ if $\Gamma = 0$, i.e. if the cylinder is not rotating. However, if the cylinder is rotating there is a finite force perpendicular to the flow, figure 10.4. This trick is often used in football and tennis!!

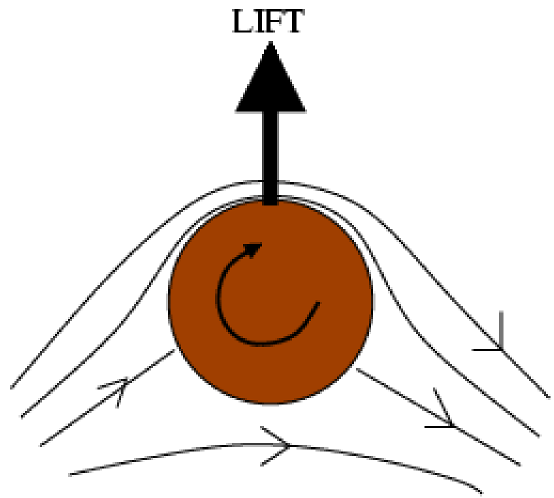


Figure 10.4: Cylinder with negative circulation