Lecture 11

Joukowski's Lift Theorem

Last lecture we found that the lift generated by a rotating cylinder is

$$F = -\rho U \Gamma.$$

In fact this expression can be applied to a much greater number of examples, as stated in *Joukowski's Lift Theorem.*

11.1 Joukowski's Lift Theorem

Consider a steady flow (uniform at ∞ with speed U in the x-direction) past a 2D body of arbitrary cross-section. Let the circulation around the body be Γ. Then the force $\mathbf{F}$ acting on this body will have the following components,

$$F_x = 0 \longleftarrow \text{(D'Alembert's Paradox)} \ ,$$
$$F_y = -\rho U \Gamma.$$

Before proving this theorem we need to consider another one.

11.2 Blasius's Theorem

Let there be a steady flow with complex potential $\chi(z)$ about some fixed body which has a closed contour C at its boundary. Then

$$F_x - iF_y = \frac{1}{2} i\rho \oint_C (\partial_z \chi)^2 dz.$$

11.3 Proof of Blasius's Theorem

Consider figure 11.1. We have,

$$|\delta \mathbf{F}| = p|dz|,$$

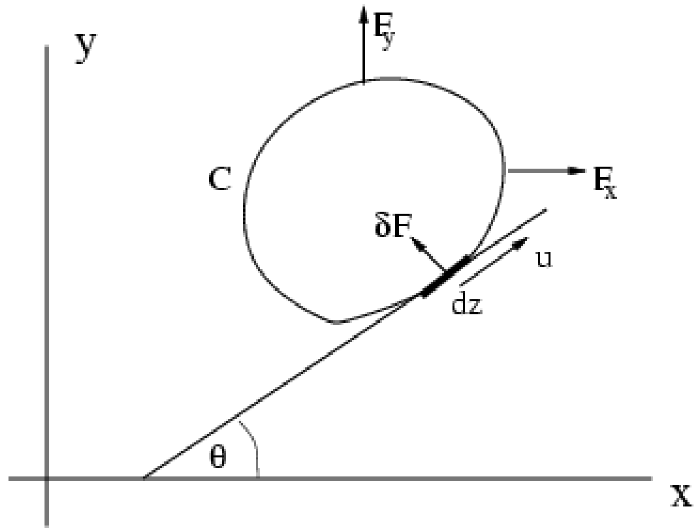

Figure 11.1: Blasius's Theorem

and

$$dz = |dz|e^{i\theta}.$$

Therefore,

$$\delta\mathbf{F} = (-\sin\theta, \cos\theta)p|dz|,$$

and

$$\delta F_x - i\delta F_y = -p(\sin\theta + i\cos\theta)|dz| = -ipe^{-i\theta}|dz|.$$

Now because C is a streamline of the flow, we can write,

$$u = |\mathbf{u}|\cos\theta, \text{ on } C,$$
$$v = |\mathbf{u}|\sin\theta, \text{ on } C,$$

where $|\mathbf{u}| = \sqrt{u^2+v^2}$. So,

$$\partial_z\chi = u - iv = |\mathbf{u}|e^{-i\theta}, \text{ on } C.$$

Taking into account this formula when using Bernoulli, we find,

$$\begin{aligned}\delta F_x - i\delta F_y &= \left(\frac{1}{2}\rho|\mathbf{u}|^2 - K\right) ie^{-i\theta}|dz| \\ &= \frac{1}{2}i\rho\left(\partial_z\chi\right)^2 e^{i\theta}|dz| - Ki(dx - idy),\end{aligned}$$

where K is a constant. Finally, replacing $|dz|\exp(i\theta)$ with dz and integrating (note the last term integrates to zero), we find

$$F_x - iF_y = \frac{1}{2}i\rho\oint_C(\partial_z\chi)^2 dz,$$

as required.

11.4 Proof of Joukowski's Lift Theorem

Consider figure 11.2. The flow around our body will tend towards the velocity U at ∞.

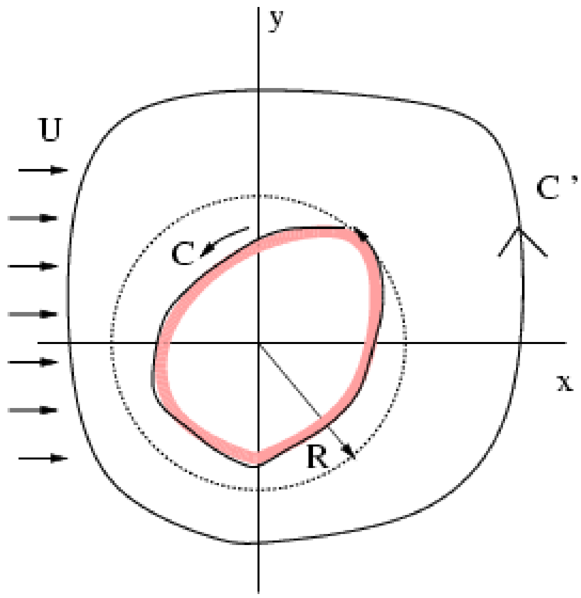

Figure 11.2: Flow Past Body

Also note that we have no singularities. Therefore, the Laurent series for $\partial_z \chi$ has the following form,

$$\partial_z \chi = U + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots . \tag{11.1}$$

This formula is valid for $|z| > R$, i.e. outside of the minimal circle enclosing the body.

Now, because there are no residues between C and C' (C' encloses the circle $|z| = R$), one can change C to C' in the Blasius Theorem and substitute in our series (11.1) for $\partial_z \chi$. Therefore, we obtain

$$\begin{aligned} F_x - iF_y &= \frac{1}{2} i\rho \oint_{C'} \left(U + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \right)^2 dz \\ &= \frac{1}{2} i\rho 2\pi i 2 U a_1 = -2\pi\rho U a_1 . \end{aligned}$$

To find a_1, we integrate (11.1) as follows,

$$\begin{aligned} 2\pi i a_1 &= \oint_{C'} \partial_z \chi dz \\ &= \oint_C \partial_z \chi dz = [\chi]_C \\ &= [\phi + i\psi]_C = [\phi]_C = \Gamma, \end{aligned}$$

where $[f]_C$ means change in f after one circuit along C. Note that $[\psi]_C = 0$ because C is a streamline and ψ is constant on it. Therefore,

$$F_x - iF_y = i\rho U \Gamma,$$

as required.