Lecture 13

Navier-Stokes Equation

13.1 The Stress Tensor

Definition - The Stress tensor T_{ij} is the i-component of stress on a surface element δS which has a normal $\mathbf{n}$ pointing in the j-direction, figure 13.1.

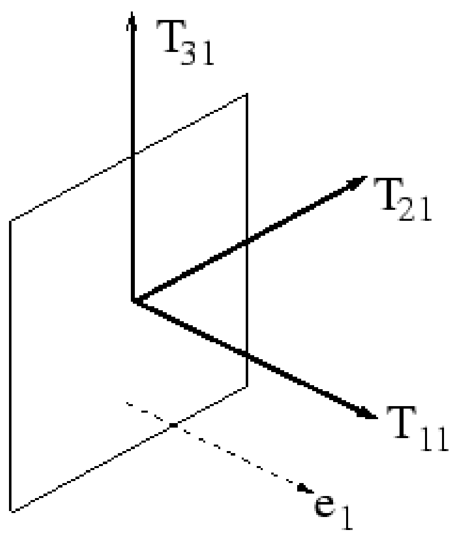

Figure 13.1: Stress Tensor

13.2 Derivation of Navier-Stokes Equation

We will now derive the Navier-Stokes Equation. As with the Euler equations we will start by considering a small fluid element (in this case a cube), figure 13.2, and examine the various forces it is subjected to. Consider the force acting in the $x - direction$,

$$\begin{aligned}\delta F_1 = \delta S[&T_{11}(x+d,y,z) - T_{11}(x,y,z) + \\ &T_{12}(x,y+d,z) - T_{12}(x,y,z) + \\ &T_{13}(x,y,z+d) - T_{13}(x,y,z)].\end{aligned}$$

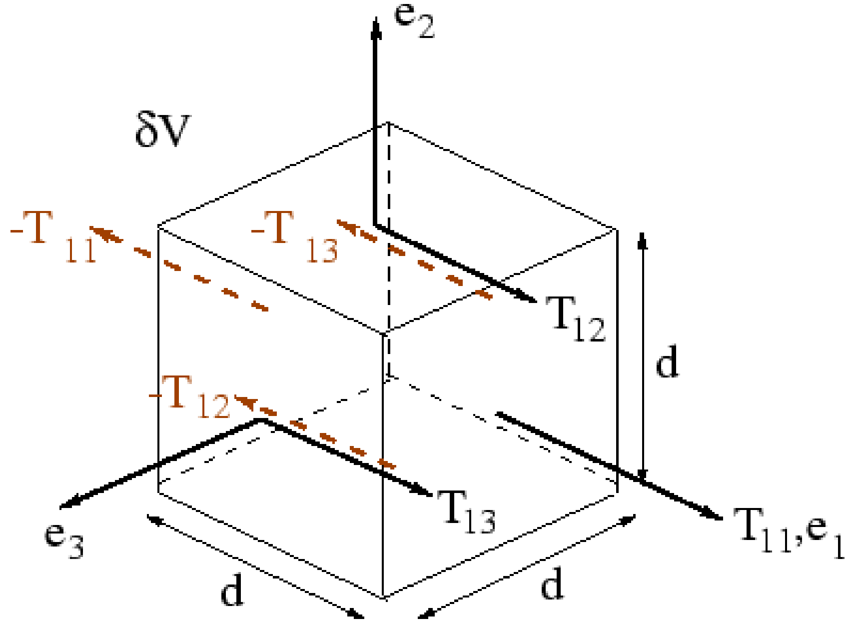

Figure 13.2: Fluid Element

However, we can re-write this as

$$\begin{aligned}\delta F_1 &\simeq d\delta S[\partial_{x_1}T_{11} + \partial_{x_2}T_{12} + \partial_{x_3}T_{13}] \\ &= \delta V \partial_{x_j} T_{ij},\end{aligned}$$

where $(x_1, x_2, x_3) = (x, y, z)$, and in the last expression we have used the summation rule of repeated indices. In general we can write,

$$\delta F_i = \frac{\delta T_{ij}}{\delta x_j} \underbrace{\triangle x_j \delta S}_{\delta V}.$$

Now, like our derivation of the Euler equations, we consider Newton's Second Law,

$$\text{mass} \cdot \text{acceleration} = \text{net force} = \delta \mathbf{F} + \text{gravity}.$$

This gives *Cauchy's Equation*

$$D_t u_i = \frac{1}{\rho}\partial_{x_j} T_{ij} + g_i,$$

which is valid for any continuous medium. Therefore, to finish our derivation we need to find T_{ij} for our fluid.

13.2.1 Case 1 - An Inviscid Fluid

We have have already seen that this case leads to the Euler Equations. Therefore, we can write

$$T_{ij} = -p\delta_{ij}.$$

13.2.2 Case 2 - A Viscid Fluid

In the viscous case we can write,

$$T_{ij} = -p\delta_{ij} + \sigma_{ij},$$

where σ_{ij} is called the *Viscous Stress Tensor*. So what form does the viscous stress tensor take? Here are a couple of key points that we must consider.

Point 1 - Galilean Invariance $\Rightarrow$ Since we have no internal friction, there are no viscous stresses if $\mathbf{u}$ is a constant. Therefore, σ_{ij} depends only $\partial_{x_i} u_j$.[1]

Point 2 - There is *no* σ_{ij} in uniform rotation, i.e. when $\mathbf{u} = \mathbf{\Omega} \times \mathbf{r}$.

Now, a combination of terms like $\partial_{x_i} u_j$ which produces a zero contribution when $\mathbf{u} = \mathbf{\Omega} \times \mathbf{r}$ is

$$\partial_{x_j} u_i + \partial_{x_i} u_j,$$

(You should check this!). Therefore, our σ_{ij} has to depend on such combinations. The most general form of σ_{ij} with these properties is,

$$\boxed{\sigma_{ij} = \eta\left(\partial_{x_j} u_i + \partial_{x_i} u_j - \tfrac{2}{3}\delta_{ij}\partial_{x_i} u_i\right) + \xi\delta_{ij}\partial_{x_i} u_i},$$

Note - We have neglected $\partial_{x_j} u_j$ nonlinear terms, assuming that they are small and that $\partial_{x_i} u_j$ terms are not too big.

The Coefficients η and ξ - The coefficients η and ξ are called the first and second viscosity coefficients respectively. To obtain them we would need to start our derivation of the fluid equations using the Boltzman kinetic equation. However, we are not going to do this! Substituting our σ_{ij} into the Cauchy equation we find,

$$D_t \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \mathbf{u} + \left(\frac{\xi}{\rho} + \frac{\nu}{3}\right)\nabla\nabla\cdot\mathbf{u},$$

where $\nu = \eta/\rho$. This equation is called the *Navier Stokes Equation.* We should not that we have not used the incompressibility condition in this derivation, therefore the NSE is valid for *both* compressible and incompressible fluids. For a *incompressible* fluid we have $\nabla \cdot \mathbf{u} = 0$ which simplifies the NSE to,

$$D_t \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \mathbf{u}.$$

In this case we omit reference to the first and second viscosities, and call ν the viscosity coefficient.

[1] We can neglect dependence on higher order derivatives if the velocity gradients aren't too strong.