Lecture 14

Incompressible NSE

We can write the incompressible NSE like this (redefining the stress tensor)

$$D_t u_i = \frac{1}{\rho}\partial_{x_j} T_{ij} + g_i,$$

where,

$$T_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

and,

$$e_{ij} = \frac{1}{2}\left(\partial_{x_j} u_i + \partial_{x_i} u_j\right),$$

which is called the *Rate-of-Strain Tensor*. (NB: the viscosity $\mu = \nu\rho$, where μ is the kinematic viscosity).

14.1 The General Deformation of a Fluid Element

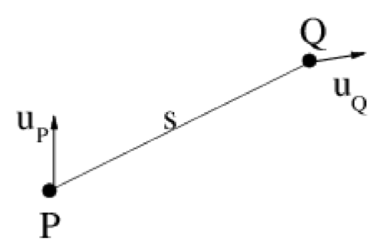

Figure 14.1: General Deformation

Consider figure 14.1. The velocity at Q can be expressed in terms of $\mathbf{u}_P$ via

$$\begin{aligned}\mathbf{u}_Q &= \mathbf{u}_P + (\mathbf{s}\cdot\nabla)\mathbf{u} \\ &= \mathbf{u}_P + \frac{1}{2}\underbrace{(\nabla\times\mathbf{u})\times\mathbf{s}}_{\text{rotation}} + \frac{1}{2}\underbrace{\nabla_{\mathbf{s}}(e_{ij}s_i s_j)}_{\text{pure strain}}.\end{aligned}$$

The rotation is at an angular velocity $\boldsymbol{\omega}/2$, while the pure strain corresponds to "stretching and squashing!".

14.2 Example - A Pure Strain Flow

Consider our previous example of a pure strain flow. Recall, that for a 2D flow the velocity profile is

$$\mathbf{u} = (\alpha x_1, -\alpha x_2, 0).$$

In this case we find that our rate-of-strain tensor takes the form,

$$e_{ij} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives a straining term of,

$$e_{ij} s_i s_j = \alpha (s_1^2 - s_2^2).$$

This gives a set hyperbolas, lines with $e_{ij} s_i s_j = const$, see figure 14.2. The velocity $\mathbf{u}$ is perpendicular to these lines.

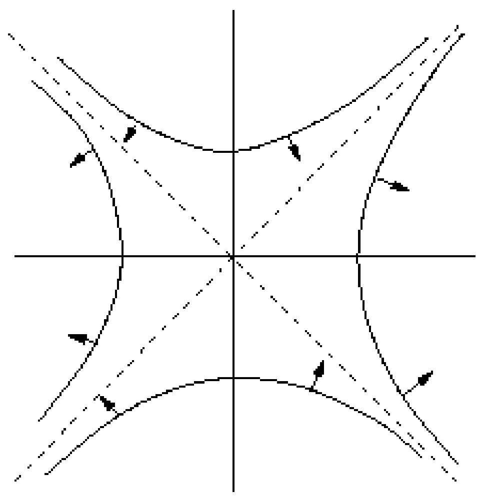

Figure 14.2: Hyperbolas

The General Case - In the general case, principal axis can always be found so that e_{ij} is diagonal. Therefore,

$$e_{ij} s_i s_j = e'_{11} s_1'^2 + e'_{22} s_2'^2 + e'_{33} s_3'^2.$$

Using this with the incompressibility condition, $e'_{11} + e'_{22} + e'_{33} = 0$, we find *surfaces* of

$$e_{ij} s_i s_j = const.$$

These are hyperboloids and the associated fluid motions are simply the 3D equivalents of those shown in figure 14.2.

Note - that in general e_{ij} depends on the coordinates x_i.

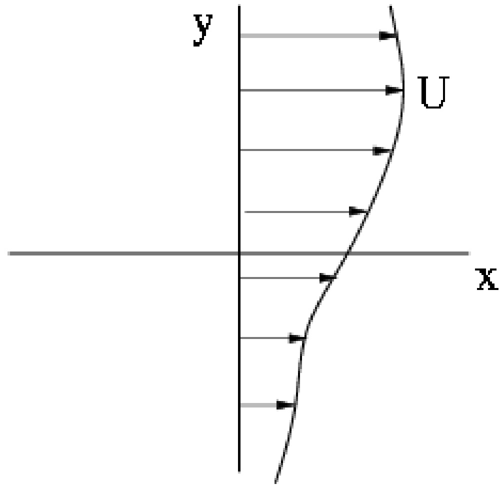

Figure 14.3: Parallel Shear Flow

14.3 Plane Parallel Shear Flow

Consider a parallel shear flow, figure 14.3, with a velocity profile,

$$\mathbf{u} = (u(y,t), 0, 0).$$

We note that the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ is satisfied automatically. The momentum equation, in component form, is

$$\begin{aligned} \partial_t u &= -\frac{1}{\rho}\partial_x p + \nu \partial^2_{yy} u, \qquad (14.1) \\ \partial_y p &= 0, \\ \partial_z p &= 0. \end{aligned}$$

Therefore, we see that p is function of x and t only. However, we also see, via equation (14.1), that $\partial_x p$ is equal to something which is independent of x. Hence, $\partial_x p$ can be a function of t only.

14.4 Stationary Flow

In the case of a stationary flow, the above argument implies that $\partial_x p$ is a constant.

$$\partial_x p = P = Const.$$

Therefore, re-arranging equation (14.1) we find,

$$\partial^2_{yy} u = \frac{P}{\rho\nu}.$$

Therefore, the velocity profile takes the form,

$$u = \frac{P}{2\rho\nu} y^2 + Ay + B.$$

We notice this expression contains two arbitrary constants, but not an arbitrary function as in the inviscid case - viscosity removes We will now consider two important examples.

14.4.1 Poiseuille Flow

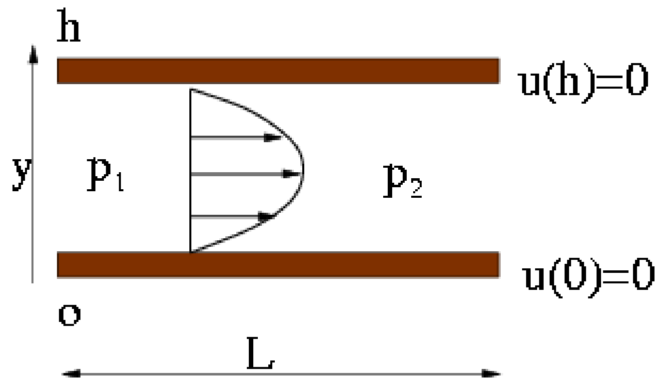

Figure 14.4: Poiseuille Flow

A Poiseuille flow is the name given to a flow between 2 fixed parallel plates, which is driven by a pressure difference $p_2 - p_1$, figure 14.4. In this case we can write our constant P as,

$$P = \partial_x p = \frac{p_2 - p_1}{L}.$$

Our BC's are $u(0) = u(h) = 0$ which gives us

$$B = 0 \text{ at } y = 0,$$

$$A = -\frac{P}{2\rho\nu} h \text{ at } y = h.$$

Therefore, our velocity profile is,

$$u = \frac{p_2 - p_1}{2L\rho\nu} y(y - h),$$

a parabolic profile. One should note that $u > 0$ when $p_2 < p_1$.

14.4.2 Couette Flow

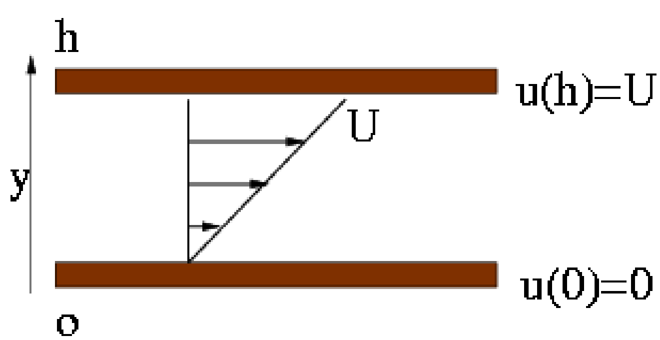

Figure 14.5: Couette Flow

A Couette flow is driven by the relative motion of two parallel plates, figure 14.5. Our constant P is now O because the flow is not driven by a pressure gradient, i.e.

$$P = \partial_x p = 0.$$

From the BC's we find our constants are,

$$A = \frac{U}{h},$$
$$B = 0.$$

Hence,

$$u = \frac{U}{h} y.$$

Remark: The solutions we've obtained for the Poiseuille and Couette flows are observed experimentally if the Reynolds Number Re is not too large. For high Re, both the flows become unstable. The instability leads to turbulence, a state where chaotic fluid motions are superimposed on the mean flow. The turbulent velocity profiles in these cases are sketched in figure 14.6. In both cases the turbulent profiles are sharpened near the boundaries and flatten in the middle.

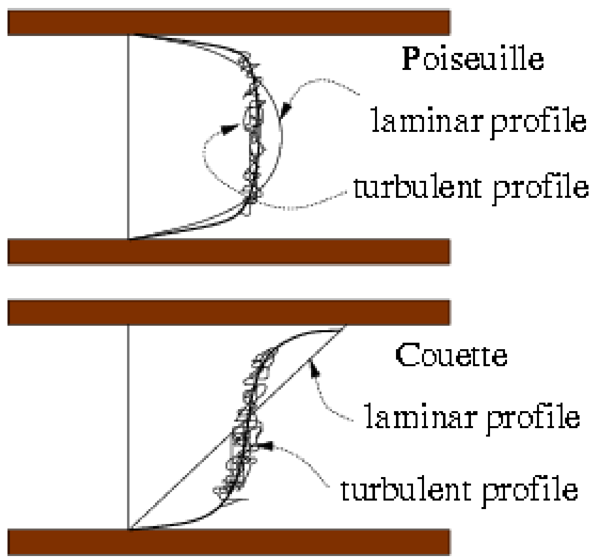

Figure 14.6: Turbulent Profiles