Lecture 15

Examples of Non-Stationary NSE solutions - Diffusion vs. Convection

15.1 Non-Stationary Flow due to an Impulsively Moved Boundary

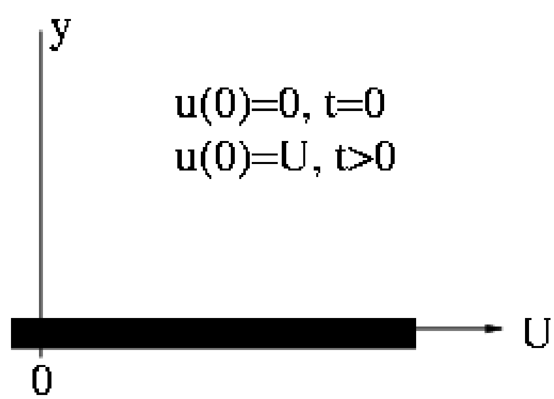

Figure 15.1: Initial Conditions

We will now move on to consider a non-stationary flow. This time we will investigate the profile generated by an impulsively moved boundary. The initial condition is shown in figure 15.1. Our governing equation now takes the form,

$$\partial_t u = \nu \partial^2_{yy} u,$$

this should be familiar, it is called the *Diffusion Equation*. There are no pressure terms in the governing equation because the flow is not driven by pressure differences. The initial conditions of our flow are as follows,

$$u(y,0) = 0, \ y > 0,$$

while the boundary conditions are,

$$u(0,t) = U, \ t > 0,$$
$$u(\infty,t) = 0, \ t > 0.$$

Observation: The diffusion equation doesn't change under the *similarity transformation*,

$$y \to \alpha y,$$
$$t \to \alpha^2 t,$$

where α is a constant. Thus, one can seek a solution in which y and t enter only in the combination $y/t^{1/2}$,

$$u = f(\eta), \; \eta = \frac{y}{(\nu t)^{1/2}},$$

where we have introduced ν in η to make it dimensionless. Using the chain rule we find the derivatives of our problem,

$$\partial_t u = f'(\eta)\partial_t \eta = -f'(\eta)\frac{y}{2\nu^{1/2}t^{3/2}},$$
$$\partial_y u = f'(\eta)\partial_y \eta = f'(\eta)\frac{\nu^{1/2}}{t^{1/2}},$$
etc ...

Substituting this into the diffusion equation gives the following ODE for $f(\eta)$,

$$f'' + \frac{1}{2}\eta f' = 0.$$

Solving for f' we find,

$$f' = Be^{-\eta^2/4}.$$

Integrating again gives,

$$f = A + B\int_0^\eta e^{-S^2/4} dS,$$

where the constants A and B are found via the BC's. In fact the BC's in terms of f are,

$$f(\infty) = 0,$$
$$f(0) = U.$$

Hence,

$$u = U\left[1 - \frac{1}{\pi^{1/2}}\int_0^\eta e^{-s^2/4} dS\right].$$

This solution is *self-similar*. That is, the solution at time $t_2 > 0$ can always be obtained from the solution at time $t_1 < t_2$ by a "stretching" along the y-axis. Figure 15.2 shows how the boundary layer has a characteristic depth of penetration of $(\nu t)^{1/2}$. One can also find the vorticity,

$$\omega = -\partial_y u = \frac{U}{(\pi\nu t)^{1/2}} e^{-y^2/4\nu t}.$$

15.2 The Return of the Vorticity Equation

Consider the NSE, after taking the curl we find,

$$\partial_t \boldsymbol{\omega} + \underbrace{(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}}_{convection} = \underbrace{(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}}_{stretching} + \underbrace{\nu\nabla^2\boldsymbol{\omega}}_{diffusion}. \tag{15.1}$$

Note - that this is the same vorticity equation which we obtained before, via the Euler equation, except now we have an extra term describing the diffusion of vorticity.

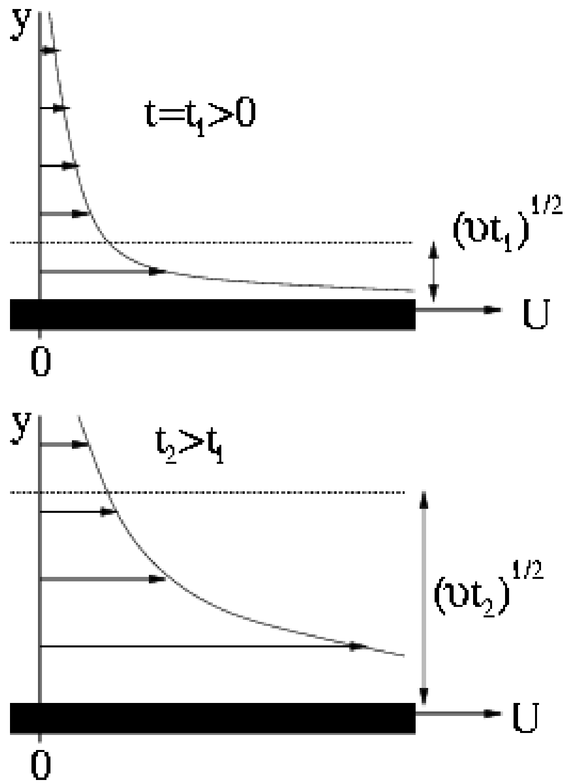

Figure 15.2: Velocity Profile at Times t_1 and t_2

15.2.1 The 2D Vorticity Equation

Recall, in two-dimensions we have $\boldsymbol{\omega} = (0, 0, \Omega)$. Therefore, equation (15.1) reduces to

$$\partial_t \Omega + (\mathbf{u} \cdot \nabla)\Omega = \nu \nabla^2 \Omega.$$

That is, there is no vorticity stretching term in 2D.

15.2.2 Circular Vortex

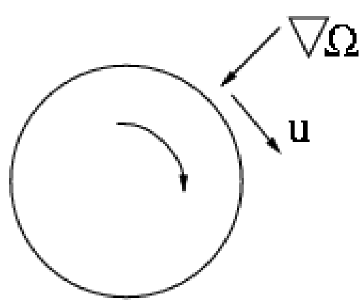

Figure 15.3: Circular Vortex

In the case of a circular vortex, figure 15.3, we make the observation that $\mathbf{u} \cdot \nabla\Omega = 0$ because $\mathbf{u} \perp \nabla\Omega$. Therefore, in this case, we have pure diffusion of vorticity,

$$\partial_t \Omega = \nu \nabla^2 \Omega, \tag{15.2}$$

with no convection or stretching terms.

15.3 Viscous Decay of a Point Vortex

You should check that

$$\Omega = \frac{\Gamma_0}{4\pi\nu t} e^{-r^2/4\nu t},$$

is a solution to equation (15.2). In doing this it's useful to write ∇^2 in terms of the radial co-ordinate r,

$$\nabla^2 = \frac{1}{r}\partial_r(r\partial_r).$$

Let's calculate the circulation of our point vortex,

$$\begin{aligned}
\Gamma &= \int \Omega dS, \\
&= 2\pi \int_0^\infty \Omega r dr, \\
&= 2\pi \int_0^\infty \frac{\Gamma}{4\pi\nu t} e^{-r^2/4\nu t} r dr, \\
&= \Gamma_0 \int_0^z e^{-z} dz, \\
&= \Gamma_0.
\end{aligned}$$

Therefore, $\Gamma = \Gamma_0$, i.e the total circulation is the same for all time. The vortex is spreading out from it's initial point shape, figure 15.4. At time $t = 0$ the vortex has vorticity,

$$\Omega(t=0) = \lim \frac{\Gamma_0}{4\pi\nu t} e^{-r^2/4\nu t} = \Gamma_0 \delta(\mathbf{x}).$$

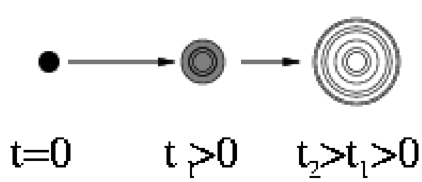

Figure 15.4: Viscous Decay of a Point Vortex

15.4 Burger's Vortex

The Burger's vortex is a special kind of vortex where all the processes - convection, stretching and diffusion - are important and, in fact, balance each other, figure 15.5. Basically, we have a vortex tube in an external strain flow. The velocity profile in this case is,

$$\begin{aligned}
u_r &= -\frac{1}{2}\alpha r, \\
u_z &= \alpha z, \\
u_\theta &= \frac{\Gamma}{2\pi r}\left(1 - e^{-\alpha r^2/4\nu}\right).
\end{aligned}$$

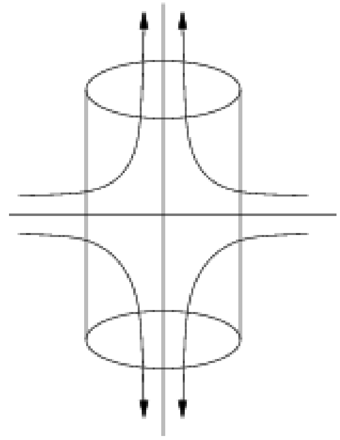

Figure 15.5: Burger's Vortex

The first two components u_r and u_z represent a potential flow, which are, in fact, a solution by themselves. The vorticity,

$$\omega = \frac{\alpha\Gamma_0}{4\pi\nu} e^{-\alpha r^2/4\nu} \mathbf{e}_z,$$

for the Burger vortex, is concentrated in a vortex core of radius order $(\nu/\alpha)^{1/2}$. The shape of this vortex is the same as our previous example, a decaying point vortex, when $t = 1/\alpha$.