Lecture 16

Very Viscous Flows

16.1 The Slow Flow Equations

Consider the steady NSE, i.e. $\partial_t \mathbf{u} = 0$,

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}.$$

Now, suppose that our Reynolds Number Re is very small,

$$Re = \frac{ul}{\nu} \ll 1.$$

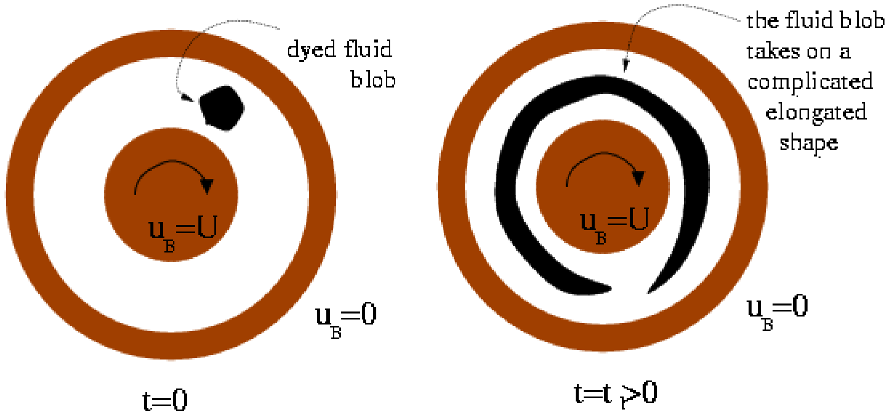
In such a case, the convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is negligible in comparison to the viscosity term $\nu \nabla^2 \mathbf{u}$. Therefore, we can re-write our NSE as,

$$\begin{aligned} 0 &= -\nabla p + \nu \rho \nabla^2 \mathbf{u}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned}$$

these are known as the *Slow Flow Equations.*

16.2 Reversibility

One important property of very viscous flows is *reversibility.* Consider a viscous fluid between two cylinders, figure 7.1, where the inner cylinder is rotating at a velocity U. A blob of dye is placed in our fluid at the initial time $t = 0$. Let the velocity on the boundary $\mathbf{u}_B = f(\mathbf{x})$. The velocity profile in our fluid $\mathbf{u}(\mathbf{x})$ is a solution of the slow flow equations, while $p(\mathbf{x})$ is the corresponding pressure. As time evolves we see our blob gets stretched out into quite a complicated shape. However, if we then reverse the flow via the reverse BC's (i.e. the turning the inner cylinder the opposite way) $\mathbf{u}_B \to -\mathbf{u}_B$. Then $-\mathbf{u}(\mathbf{x})$ is a solution of the slow flow equations with a corresponding pressure $-p(\mathbf{x})$. In fact, what we find is that the blob will return to its initial shape, figure 7.2. This can only be observed in very very low Re flows.



(a) (b)

Figure 16.1: Reversible Flow

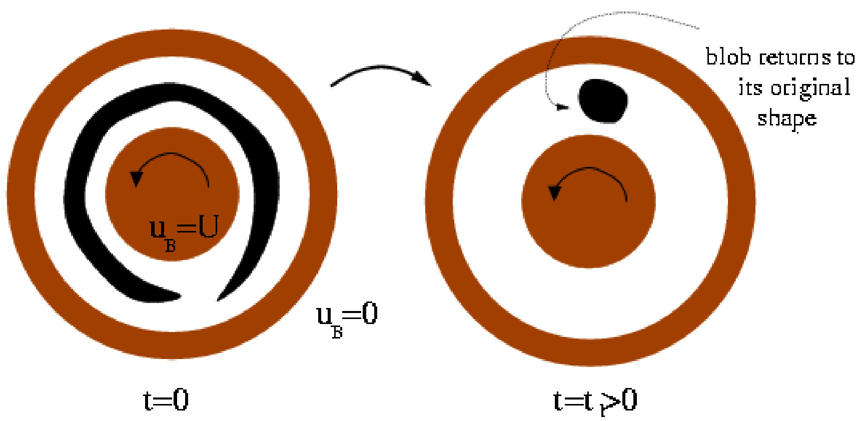


Figure 16.2: Reversible Flow with Reverse BC's

16.3 An Adhesive Problem and Spilt Beer

Consider a thin circular disc separated from a boundary (table) by a thin film of water (or beer), figure 16.3. We will try to pull the disk off the boundary. (i.e. we are really investigating why a pint glass appears to stick to the table when it is sitting a thin puddle of spilt beer!). From the geometry of the problem we can split the fluid velocity into two components,

$$\mathbf{u} = u_r(r,z,t)\mathbf{e}_r + u_z(r,z,t)\mathbf{e}_z,$$

We can use the continuity equation to get more of an insight,

$$\frac{1}{r}\partial_r(ru_r) + \partial_z u_z = 0.$$

From figure 16.3 we see that,

$$\begin{aligned} \partial_r a_r &\sim u_r/a, \\ \partial_z a_z &\sim u_z/h. \end{aligned} \tag{16.1}$$

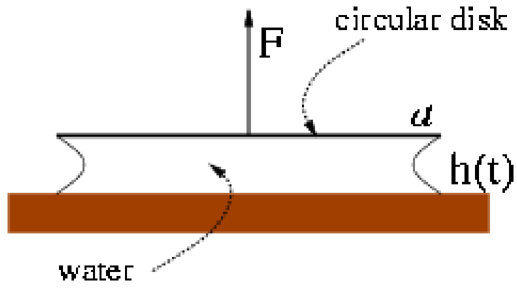


Figure 16.3: Adhesive Problem

Therefore, using the continuity equation and the fact that $h \ll a$, this tell us that

$$u_r \sim \frac{a}{h} u_z \gg u_z.$$

i.e. the velocity field is mainly in the r-direction (although is has its greatest variation in z).

We can use these order-estimate type arguments for the slow flow equations. Again, because $h \ll a$, the second order derivatives are $\partial_{zz} u_{r,z} \sim \frac{1}{h^2} u_{r,z} \gg \frac{1}{a^2} u_{r,z} \sim \partial_{rr} u_{r,z}$. The slow flow equations in this example are,

$$\partial_r p = \nu\rho \partial_{zz} u_r,$$
$$\partial_z p = \nu\rho \partial_{zz} u_z.$$

Now as $\partial_r p \sim \frac{\nu\rho}{h^2} u_r \sim \frac{\nu\rho}{h^2}\frac{a}{h} u_z \gg \frac{\nu\rho}{h^2} u_z \sim \nu\rho\partial_{zz} u_z$. This tells us that $\partial_r p \gg \partial_z p$, i.e. the pressure varies the most in the r-direction. (NB: this is opposite to the result obtained for u above, which has the greatest variation in the z-direction, (16.1)). The pressure p is projected through the thin layer (in a similar manner to the important BL problem).

From the radial component of the slow flow equations we find that the pressure is a function of r and t only, $p = p(r,t)$. We are now in a position to integrate the equations and apply the BC's,

$$u_r = 0 \text{ at } z = 0 \text{ and } z = h(t).$$

The resulting expression for the radial velocity is,

$$u_r = \frac{1}{2\nu\rho} z(z-h)\partial_r p.$$

To find the z-component of the velocity we use the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ (see above). Integrating over z, and using the BC $u_z = 0$ on $z = 0$, we find,

$$u_z = -\frac{1}{2\nu\rho r}\partial_r(r\partial_r p)\left(\frac{z^3}{3} - \frac{hz^2}{2}\right).$$

However, we also have another BC,

$$u_z = \partial_t h \text{ at } z = h(t).$$

This gives us an equation for p_r,

$$\partial_r(r\partial_r p) = \frac{12\nu\rho r}{h^3}\frac{dh}{dt}.$$

Integrating over r, we obtain,

$$\partial_r p = \frac{6\nu\rho r}{h^3}\frac{dh}{dt} + \frac{C(t)}{r}.$$

We can set $C(t) = 0$ to avoid an unphysical ∞ at $r = 0$. Integrating again we find,

$$p = \frac{3\nu\rho}{h^3}\frac{dh}{dt}r^2 + D(t),$$

where $D(t)$ is found using the condition $p = p_0$ at $r = a$ (here p_0 is the atmospheric pressure). This gives us,

$$p - p_0 = \frac{3\nu\rho}{h^3}\frac{dh}{dt}(r^2 - a^2).$$

To find the force on the disk, one must integrate the pressure difference over the disk area,

$$F = \int_0^{2\pi}\int_0^a (p - p_0) r dr d\theta = \frac{3\pi}{2}\frac{\nu\rho a^4}{h^3}\frac{dh}{dt}.$$

Hence, we see that F is very strong when h is small.