# Lecture 17

# Boundary Layer Theory

## 17.1 Deriving the BL Equations

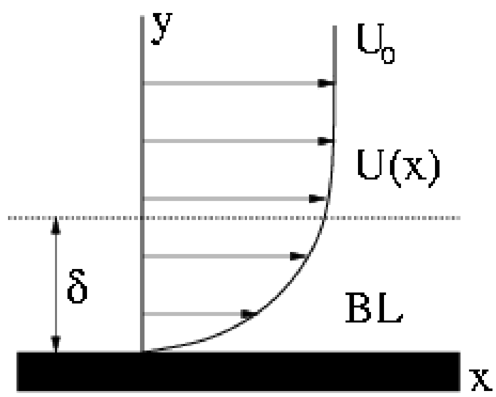


Figure 17.1: Boundary Layer Theory

Consider a fluid flow near a boundary, figure 17.1. We would like to derive a set of Boundary Layer (BL) equations to describe the fluid motion in such a region. Let $\delta$ be the thickness of the Boundary Layer (BL), and $L$ the characteristic size of our body (or curvature radius). We will assume that $\delta \ll L$, (which is true, as we will see later, if $Re >> 1$). We also see from our figure that,

$$|\partial_y u| \sim \frac{u}{\delta} \gg \frac{u}{L} \sim |\partial_x u|.$$

Writing out the stationary NSE's in component form we have,

$$\begin{aligned}
u\partial_x u + v\partial_y u &= -\frac{1}{\rho}\partial_x p + \nu(\partial_{xx}u + \partial_{yy}u), \\
u\partial_x v + v\partial_y v &= -\frac{1}{\rho}\partial_y p + \nu(\partial_{xx}v + \partial_{yy}v), \\
\partial_x u + \partial_y v &= 0.
\end{aligned}$$

From the last equation, the continuity equation, we see that,

$$|\partial_y v| \sim |\partial_x u| \sim \frac{U_0}{L} \to v \sim \frac{U_o\delta}{L}.$$

This means that $v \ll u$, which implies that in our momentum equation,

$$|\partial_y p| \ll |\partial_x p|.$$

In other words, $p$ is approximately a function of $x$ only.
(NB: notice the similarity to the adhesive problem).

**Prandtl** - "*The pressure distribution of the free fluid will be impressed on the transition layer*".

Continuing, we also note that,

$$\partial_{xx} u = O\left(\frac{U_0}{L^2}\right),$$
$$\partial_{yy} u = O\left(\frac{U_0}{\delta^2}\right).$$

Taking into account all these various estimates, we can write our final 2D steady BL equations as

$$u\partial_x u + v\partial_y u = -\frac{1}{\rho}\partial_x p + \nu\partial_{yy} u,$$
$$\partial_x u + \partial_y v = 0,$$

where $p$ is a function of $x$ only.

One BC of our problem is that $u = v = 0$ at $y = 0$. Comparing nonlinearity and viscosity in the BL equation, we can find the thickness $\delta$.

$$\frac{U_0}{L} \sim \frac{\nu U_0}{\delta^2}.$$

That is,

$$\frac{\delta}{L} = O(Re^{-1/2}).$$

Therefore, our assumption that $\delta \ll L$ is correct if $Re \gg 1$. It is assumed also that we have no separation.

## 17.2 BC's

The pressure is given by the outer "*inviscid*" solution $\mathbf{U}(\mathbf{x})$ and can be found from Bernoulli's equation, $p + \frac{1}{2}\rho U^2 = const$, so

$$-\frac{1}{\rho}\partial_x p = U\partial_x U.$$

i.e. $p$ is totally determined by the outer flow solution. Here $U = U(x)$ is the "*slip*" velocity of the inviscid solution (tangential to the boundary). Another BC should be that

$$u \to U(x) \text{ as } \frac{y}{\delta} \to \infty.$$

In other words, the outer solution has to be found by solving the Euler equation (ignoring $\nu$) with the *free-slip* BC's (i.e. $u_\perp = 0$ and $u_{||}$ is arbitrary at the boundary). This solution gives $u_{||}$ at the boundary which has to be used as $U(x)$ for the BL equations.

## 17.3 A Semi-infinite Plate

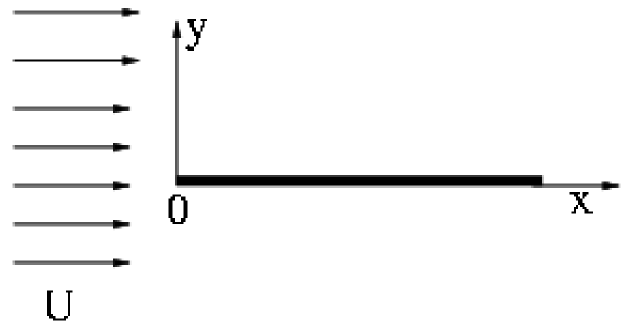


Figure 17.2: Semi-infinite Plate

We will consider the flow over a semi-infinite flat plate, figure 17.2. Recall, from the previous sections we have the 2D steady boundary layer equations,

$$
\begin{aligned}
u\partial_x u + v\partial_y u &= -\frac{1}{\rho}\partial_x p + \nu\partial_{yy}u,\\
\partial_x u + \partial_y v &= 0.
\end{aligned}
$$

The inviscid solution to our problem is a uniform flow with velocity $U$. This means that the pressure gradient term, in the above equations, is zero. We will seek a similarity solution, $u = u(\eta)$, where

$$\eta = \frac{y}{g(x)}.$$

We can also introduce a streamfunction,

$$
\begin{aligned}
u &= \partial_y\psi,\\
v &= -\partial_x\psi.
\end{aligned}
$$

Therefore, writing $u = Uh(\eta)$ allows us to integrate $u = \partial_y\psi$,

$$\psi = Ug(x)\int_0^\eta h(s)ds + K(x).$$

Now, we know the plate will be a streamline $\psi = const$. We can, therefore, choose $\psi = 0$ since it is only the derivatives of $\psi$ that are important.

$$\psi = 0 \text{ at } \eta = 0 \Rightarrow K(x) = 0.$$

It is convenient now to write $\psi$ as,

$$\psi = Ug(x)f(\eta) \text{ with } f(0) = 0.$$

Then,

$$
\begin{aligned}
u &= Uf'(\eta),\\
\partial_x u &= Uf''(\eta)\partial_x\eta = Uf''\left(\frac{-y}{g^2}\right)g',\\
\partial_y u &= Uf''(\eta)\partial_y\eta = Uf''\frac{1}{g},
\end{aligned}
$$

and,

$$
\begin{aligned}
v = -\partial_x \psi &= -U(g'f + gf'\partial_x \eta), \\
&= -U\left(g'f - \frac{y}{g}f'g'\right), \\
&= U(\eta f' - f)g'.
\end{aligned}
$$

Substituting these expressions for $u$ and $v$ into the first BL equation we find,

$$
-U^2 f'f''\frac{y}{g^2}g' + U^3(\eta f' - f)g'\frac{f''}{g} = \nu U\frac{f'''}{g^2}.
$$

Or,

$$
f''' + \frac{Ugg'}{\nu}ff'' = 0.
$$

We can now choose $g(x)$ to make this equation more simple,

$$
gg' = \frac{\nu}{U} \to \frac{1}{2}g^2 = \frac{\nu x}{U} + d.
$$

So what is $d$? Well, some quantities, like

$$
\partial_y u = U\frac{f''}{g},
$$

become $\infty$ when $g(x) = 0$. The point at which this happens should be on the leading edge of $x = 0$, hence we let $d = 0$ and

$$
\psi = (2\nu Ux)^{1/2}f(\eta).
$$

Therefore, we have to solve the equation,

$$
f''' + ff'' = 0,
$$

with the BC's,

$$
\begin{aligned}
&f(0) = 0 \text{ from definition of } f(x), \\
&f'(0) = 0 \text{ from } u(0) = 0, \\
&f'(\infty) = 1 \text{ from } u(\infty) = U.
\end{aligned}
$$

This problem is solved numerically. Then, $f'$ gives the BL velocity profile, figure 17.3. The characteristic thickness of the boundary is,

$$
\delta = O\left(\frac{\nu x}{U}\right)^{1/2}.
$$

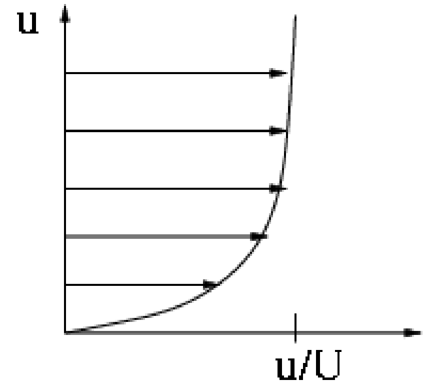


Figure 17.3: BL profile

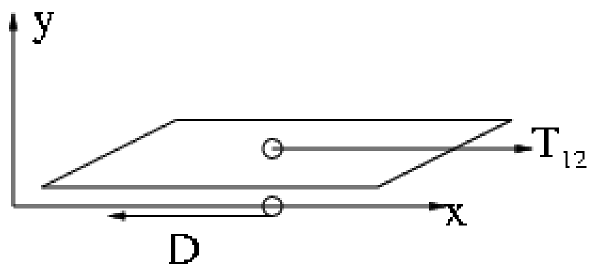


Figure 17.4: Drag on a Plate

## 17.4 The Drag on a Plate

Consider a plate, parallel to a uniform flow in the $x$ direction, figure 17.4. Then, we can write down an expression for the flux of momentum in the $x$ direction. We'll call this flux $t_x$.

$$\begin{aligned} t_x &= \nu\rho(\partial_x v + \partial_y u) \\ &\cong \nu\rho\partial_y u, \\ &= \mu U \left(\frac{U}{2\nu x}\right)^{1/2} f''(0). \end{aligned}$$

If the plate is of finite length $L$, then the drag $D$ is

$$\begin{aligned} D &= 2\int_0^L t_x dx, \\ &= 2\sqrt{2} f''(0) \rho U^2 L Re^{-1/2}. \end{aligned}$$

This is the drag per unit length, where $Re$ is the Reynold's Number

$$Re = \frac{UL}{\nu}.$$

Numerical experiments give $f''(0) = 0.47$.