Lecture 18

Taylor-Proudman Theorem and Ekman's BL

18.1 Rotating Flows with BL's

Consider a fluid between two rotating disks at $z = 0$ and $z = L$ respectively, figure 18.1. The angular velocity of the lower disk is Ω and of the upper disk is $\Omega(1+\epsilon)$. It is

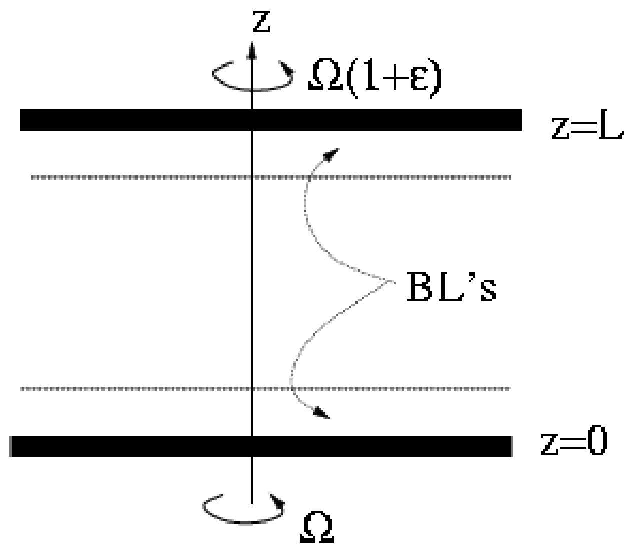

Figure 18.1:

convenient to write the NSE in a rotating frame of reference, which has angular velocity Ω. Hence,

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}.$$

The term $2\boldsymbol{\Omega}\times\mathbf{u}$ originates from the Coriolis Force in a rotating frame, while $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x})$ represents a centrifugal term. However, this centrifugal term can be re-written as,

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) = -\nabla\left(\frac{1}{2}(\boldsymbol{\Omega}\times\mathbf{x})^2\right),$$

and hence may be included with the pressure gradient term, resulting in a *"reduced pressure"* p_R where,

$$p_R = p - \frac{1}{2}\rho(\mathbf{\Omega} \times \mathbf{x})^2.$$

We are interested in flows with $\mathbf{u}$ much smaller than the rotation speed, that is,

$$\frac{u}{\Omega L} \ll 1. \tag{18.1}$$

We will now use some order estimates to determine the importance of various terms in our equations. Let us compare the convective and Coriolis terms. We find,

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \sim O\left(\frac{u^2}{l}\right),$$

where l is a characteristic length-scale and $l_{min} = \delta$ (δ as usual being the thickness of the BL). In contrast the Coriolis term gives,

$$|2\mathbf{\Omega} \times \mathbf{u}| \sim O(\Omega u).$$

Therefore, we can neglect the convective term if,

$$\frac{u}{\Omega l} \ll 1.$$

We should notice that this condition is stronger than (18.1) since $l < L$. We will assume that this is valid, (later we will find when this is the case). Therefore the NSE's reduce to,

$$\begin{aligned}
\partial_t \mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} &= -\frac{1}{\rho}\nabla p_R + \nu\nabla^2\mathbf{u}, \\
\nabla \cdot \mathbf{u} &= 0.
\end{aligned}$$

18.1.1 Steady Inviscid Flow and the Taylor Proudman Theorem

For the inner flow (outside the BL's) we can neglect ν. Writing our equations of motion in components we find, for the stationary case, that

$$\begin{aligned}
-2\Omega v &= -\frac{1}{\rho}\partial_x p_R, \\
2\Omega u &= -\frac{1}{\rho}\partial_y p_R, \\
0 &= -\frac{1}{\rho}\partial_z p_R, \\
0 &= \partial_x u + \partial_y v + \partial_z w.
\end{aligned}$$

The third equation, here, tells us that p_R is independent of z. Using this fact, we see that the first two equations tell us that both u and v are independent z too. Finally, substituting the first two equations into the last, we find that w is also independent of z. Hence, $\mathbf{u}$ is independent of z. This is the *Taylor-Proudman Theorem.*

18.1.2 The Ekman BL

Consider the BL on $z = 0$. Variations of $\mathbf{u}$ with z are much more rapid than those with x or y. This implies that we can neglect the ∂_{xx} and ∂_{yy} parts of the viscosity terms,

$$
\begin{aligned}
-2\Omega v &= -\frac{1}{\rho}\partial_x p_R + \nu \partial_{zz} u, \\
2\Omega u &= -\frac{1}{\rho}\partial_y p_R + \nu \partial_{zz} v, \\
0 &= -\frac{1}{\rho}\partial_z p_R + \nu \partial_{zz} w, \\
0 &= \partial_x u + \partial_y v + \partial_z w.
\end{aligned}
$$

From the fourth equation we see that $w \ll u, v$. This in turn implies that $\partial_z p_R \ll \partial_x p_R$ and, from $\partial_y p_R$, that p is a function of x and y only. Thus, p_R is determined by the inviscid solution outside of the BL (c.f. the BL of the flat plate in our previous example).

Substituting p_R from the inviscid solution we find,

$$
\begin{aligned}
-2\Omega(v - v_I) &= \nu \partial_{zz} u, \\
2\Omega(u - u_I) &= \nu \partial_{zz} v,
\end{aligned}
$$

where u_I and v_I are the "*inviscid*" velocities at (independent of z). Multiplying the second of these equations by i and adding this to the first, we can combine them into the form,

$$\nu \partial_{zz} f = 2\Omega i f,$$

where $f = u - u_I + i(v - v_I)$. The general solution of this equation is,

$$f = Ae^{-(1+i)z_*} + Be^{(1+i)z_*},$$

where,

$$z_* = \left(\frac{\Omega}{\nu}\right)^{1/2} z,$$

and A and B are arbitrary functions of x and y. To match this solution to the interior flow we require,

$$f \to 0 \text{ as } z_* \to \infty \Rightarrow B = 0,$$

We also have the BC's $u = v = 0$ at $z = 0$, which gives A. Therefore,

$$f = -(u_I + iv_I)e^{-(1+i)z_*}.$$

Hence,

$$
\begin{aligned}
u &= u_I - e^{-z_*}(u_I \cos z_* + v_I \sin z_*), \\
v &= v_I - e^{-z_*}(v_I \cos z_* - u_I \sin z_*).
\end{aligned}
$$

This is known as the *Ekman Boundary Layer.* The BL thickness δ is

$$\delta = \left(\frac{\nu}{\Omega}\right)^{1/2}.$$

Note that this is the only combination of the available parameters which have units of length.

18.1.3 Finding the Velocity w

Let us find w.. From the continuity equation we know that

$$\left(\frac{\Omega}{\nu}\right)^{1/2} \partial_{z_*} w = \partial_z w = -(\partial_x u + \partial_y v),$$
$$= (\partial_x v_I - \partial_y u_I) e^{-z_*} \sin z_* - (\partial_x u_I + \partial_y v_I)(1 - e^{-z_*} \cos z_*),$$

and the last term here disappears again via the continuity equation, $\partial_x u_I + \partial_y v_I = 0$. Integrating the above equation we find,

$$w_I = w\,|_{z=\infty} = \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{1/2} (\partial_x v_I - \partial_y u_I), \tag{18.2}$$
$$= \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{1/2} \omega_I, \tag{18.3}$$

where ω_I is the z-component of vorticity in the internal flow, measured in the rotating frame. If the boundary is rotating with angular velocity Ω_B then

$$w_I = \left(\frac{\nu}{\Omega}\right)^{1/2} \left(\frac{1}{2}\omega_I - \Omega_B\right),$$

where w_I is the interior vorticity measured in the laboratory (non-rotating frame). If Ω_T is the angular velocity of the upper boundary ($z = L$) then

$$w_I = \left(\frac{\nu}{\Omega}\right)^{1/2} \left(\Omega_T - \frac{1}{2}\Omega_I\right).$$

18.1.4 Determination of the Interior Flow

We need to match the w_I found near the "top" boundary to the one found near the "bottom" boundary. Hence, we require,

$$\frac{1}{2}\omega_I - \Omega_B = \Omega_T - \frac{1}{2}\omega_I,$$
$$\Rightarrow \omega_I = \Omega_B + \Omega_T.$$

In the frame rotating with the lower plate:

$$\omega_I = \partial_x v_I - \partial_y u_I = \omega_I' - 2\Omega_B = \Omega\epsilon, \tag{18.4}$$

where $\Omega\epsilon = \Omega_T - \Omega_B$. In cylindrical co-ordinates we have,

$$\frac{1}{r}\partial_r(r u_{\theta I}) = \Omega\epsilon.$$

Integrating this equation and taking into account that the solution is finite at $r = 0$,

$$u_{\theta I} = \frac{1}{2}\Omega\epsilon r.$$

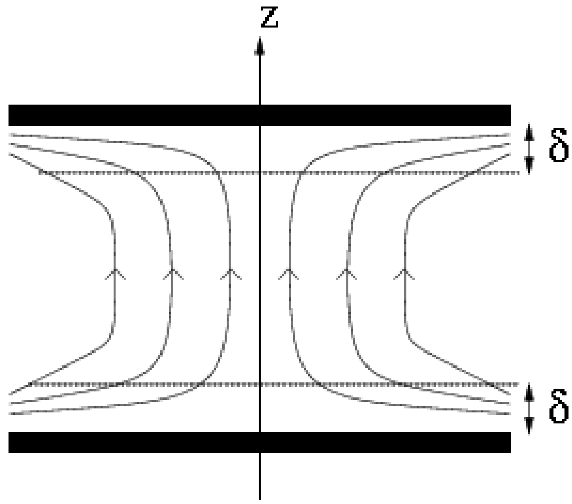

Figure 18.2: Flow between Rotating Discs

The interior inviscid fluid is uniformly rotating with an angular velocity. This velocity is the mean value of the velocities found at our two boundaries. The solution for ω_I can be found by substituting (18.4) into (18.2).

$$\omega_I = \frac{1}{2}(\nu\Omega)^{1/2}\epsilon.$$

While, from the incompressibility condition we find $u_{rI} = 0$. Therefore, our flow takes the form of that sketched in figure 18.2, where δ is a lot less than L. Finally, we return to the validity of our analysis. Remember we assumed that,

$$\frac{u}{\Omega\delta} \ll 1.$$

Now, we can substitute for u, this gives

$$\frac{\Omega\epsilon L}{\Omega\delta} \ll 1.$$

Hence,

$$\epsilon L \ll \delta \ll L \Rightarrow \epsilon \ll 1.$$

i.e. there can't be a big difference between the angular velocities of our two rotating discs.