Lecture 19

Waves in Rotating Fluids

Today we will start considering waves in fluids. Actually, the material from our previous lecture provides a smooth transition to the particular case of waves in a rotating fluid. Consider an incompressible fluid under uniform rotation. Waves in such a fluid are due to the Coriolis force (which occurs under rotation). We mentioned, at the beginning of the last lecture, that the NSE in this case (in rotating co-ordinates) takes the form,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u},$$

where we now have extra terms for Coriolis and centrifugal forces. As in the previous lecture, we can re-write the centrifugal term and include it with the exist pressure gradient. This leads to a "*reduced pressure*", p_R. We will consider a high Reynolds number flow away from the boundaries where viscosity is unimportant. In this case the above equation reduces to the Euler equation. Therefore, our equation of motion now takes the form,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p_R,$$

where

$$p_R = p - \frac{1}{2}\rho(\boldsymbol{\Omega} \times \mathbf{x})^2.$$

The Coriolis force is $2\boldsymbol{\Omega} \times \mathbf{u}$, and occurs only when the fluid has a motion relative to the rotating co-ordinates. The continuity equation, for an incompressible fluid, is unchanged and remains $\nabla \cdot \mathbf{u} = 0$.

If we consider small amplitude waves, we can linearise our equations and hence neglect the quadratic non-linear convection term. Hence, the momentum equation becomes,

$$\partial_t \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p'_R,$$

where p'_R is the variable part of the (reduced) pressure in the wave and ρ is a constant. The pressure term can be eliminated from the equation by taking the curl, since recall from vector calculus $\nabla \times \nabla \phi = 0$. Therefore, we find

$$\partial_t \nabla \times \mathbf{u} + \nabla \times (2\boldsymbol{\Omega} \times \mathbf{u}) = 0.$$

However, since the fluid is incompressible, we can reformulate the second term via,

$$\nabla \times (\boldsymbol{\Omega} \times \mathbf{u}) = \boldsymbol{\Omega}\nabla \cdot \mathbf{u} - (\boldsymbol{\Omega} \cdot \nabla)\mathbf{u} = -(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u}.$$

So, taking the z-axis to be in the same direction as Ω (i.e. along the axis of rotation), our equation becomes,

$$\partial_t \nabla \times \mathbf{u} = 2\Omega \partial_z \mathbf{u}.$$

Now, as is traditional in wave theory, we seek a plane wave solution of the form,

$$\mathbf{u} = \mathbf{A} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)},$$

which, because of the incompressibility condition, satisfies the transversality condition $\mathbf{k} \cdot \mathbf{A} = 0$. Substituting this solution into our equation we find,

$$\omega \mathbf{k} \times \mathbf{u} = 2i\Omega k_z \mathbf{u},$$

where ω is the wave frequency. The dispersion relation for these waves is found by eliminating $\mathbf{u}$ from the above vector equation. Vector multiplication on both sides by $\mathbf{k}$ gives,

$$-\omega k^2 \mathbf{u} = 2i\Omega k_z \mathbf{k} \times \mathbf{u},$$

and a comparison of these two equations yields the dependence of ω on $\mathbf{k}$,

$$\omega = \pm 2\Omega \frac{k_z}{k} = \pm 2\Omega \cos\theta,$$

where θ is the angle between $\mathbf{\Omega}$ and $\mathbf{k}$.