Lecture 1

Module Basics and Revision

1.1 What is this course about?

We are interested in finding and understanding solutions to the Navier-Stokes equations (NSE). A central example is that of an *incompressible fluid*:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

where,

$\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u, v, w)$ is the velocity;
$\mathbf{x} = (x, y, z)$ is the 3D coordinate;
$p = p(\mathbf{x}, t)$ is the pressure field;
ρ is the density;
$\mathbf{f}$ is a force (e.g. gravity).

The velocity $\mathbf{u}$ satisfies boundary conditions (BC's) which will be applied at a solid surface/boundary (e.g. $\mathbf{u} = \mathbf{u}_b$), at a free surface or at infinity. The BC at the surface of a fixed body is $\mathbf{u}_b = 0$. BC's may also be periodic, this is often the case in numeric models.

1.2 Limits and Special Cases of NSE

2D NSE $(\mathbf{x} = (x, y, 0), \mathbf{u} = (u, v, 0))$
This is commonly used in Atmospheric Modelling.

Euler's Equation $(\nu = 0)$
This is the Inviscid limit of NSE's.

Irrotational Flow $(\nabla \times \mathbf{u} = 0)$
We can introduce a velocity potential ϕ via $\mathbf{u} = \nabla\phi(\mathbf{x}, t)$. Aerodynamics are often modelled with potential flows.

Stationary Flow $(\partial_t \mathbf{u} = 0)$

1.3 Generalisations

Compressibility Density ρ not constant.

The density becomes a function of space and time $\rho = \rho(\mathbf{x}, t)$. When the pressure $p = p(\rho)$ the fluid is said to be barotropic. We can neglect compressibility when the Mach number $Ma = \frac{|u|}{c} \ll 1$.

Stratification e.g Ocean Buoyancy

We can still treat the fluid as incompressible, but $\rho = \rho(\mathbf{x})$.

Rotation e.g the Coriolis Force.

Thermal Effects e.g. salinity, moisture.

1.4 Vector Analysis Revision

Fluid equations, in general, are vector PDE's, therefore vector calculus techniques are essential.

1.4.1 Vector Identities

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c}) \cdot \mathbf{b} - (\mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{a}, \tag{1.3}$$

$$\nabla \times \nabla \phi = 0, \tag{1.4}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0, \tag{1.5}$$

$$\nabla \cdot (\phi \mathbf{F}) = \phi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \phi, \tag{1.6}$$

$$\nabla \times (\phi \mathbf{F}) = \phi \nabla \times \mathbf{F} + (\nabla \phi) \times \mathbf{F}, \tag{1.7}$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}), \tag{1.8}$$

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}, \tag{1.9}$$

$$(\mathbf{F} \cdot \nabla)\mathbf{F} = \mathbf{F} \times (\nabla \times \mathbf{F}) + \nabla(\frac{1}{2}\mathbf{F}^2), \tag{1.10}$$

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}). \tag{1.11}$$

1.4.2 Gauss's (Divergence) Theorem

$$\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_V \nabla \cdot \mathbf{F}\, dV \tag{1.12}$$
$$\Longrightarrow \int_S \phi \mathbf{n}\, dS = \int_V \nabla \phi\, dV$$
$$or \Longrightarrow \int_S \mathbf{F} \times \mathbf{n}\, dS = -\int_V \nabla \times \mathbf{F}\, dV$$
$$or \Longrightarrow \int_S \mathbf{n} \cdot \nabla \phi\, dS = \int \nabla^2 \phi\, dV.$$

1.4.3 Stokes's Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \tag{1.13}$$
$$\Longrightarrow \oint_C \mathbf{u} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = \int_S \omega \cdot \mathbf{n} dS$$
$$or \Longrightarrow \int_C \phi\, d\mathbf{x} = -\int_S (\nabla \phi) \times \mathbf{n}\, dS.$$