Lecture 20

An Introduction to Waves

20.1 Wave Basics

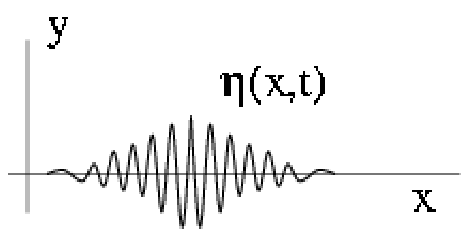

Figure 20.1: A Group of Surface Waves

In this section we will discuss some of the fundamentals of wave theory. (See also the handout on Fourier series and Fourier integrals). We'll outline what we mean by wavenumber, phase and group velocities, dispersion etc. Consider a group of surface waves moving across a pond, figure 20.1. We will assume in our system that the wavelength of out waves are much shorter than the depth of the pond. This is known as the deep water limit. In such a scenario we can make the following observations.

Observation 1 - In this example, each wave-crest travels faster than the group as a whole.

Observation 2 - New crests appear to be created at the back of the group and disappear at the front.

These effects are due a process known as *dispersion*.

Dispersion - Different Fourier components (modes) that make up a general disturbance propagate at different speeds (depending on their wavelength).

As an example we will consider the simple case of a harmonic surface wave,

$$\eta = A\cos(kx - \omega t),$$

where A is the wave amplitude, k the wavenumber (a wavevector in 2D and 3D) and ω is the wave frequency (not to be confused with vorticity!). You can think of the wavenumber as another way of specifying the wavelength λ of a given wave. In fact, we have the very simple relationship that,

$$k = \frac{2\pi}{\lambda}.$$

This is why, in spectral (or wavenumber) space, large-scales will have small k and small-scales large k.

The *Dispersion Relation* is a relationship between a waves frequency and wavenumber, i.e. $\omega = \omega(k)$. Now, the wave speed or *Phase Velocity* is defined as,

$$c_{ph} = \frac{\omega}{k}.$$

For deep water waves, we will find in the next lecture that, the dispersion relationship is,

$$\omega = (gk)^{1/2}.$$

This gives a phase velocity of,

$$c_{ph} = \left(\frac{g}{k}\right)^{1/2}.$$

The phase velocity is the speed of the crests which make up our group of waves on the pond. From this simple relation we see that, in deep water, the longer wavelength waves travel faster. The *Group Velocity* is defined as,

$$c_g = \frac{d\omega}{dk},$$

and not surprisingly this is the speed of the group of waves as a whole. For our deep water case we find,

$$c_g = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2} = \frac{1}{2}c_{ph}.$$

So in this case, we see that the individual waves, which make up the group (or wavepacket), travel twice as fast as the group as a whole. Thus explaining the observations we made at the beginning of this section.

One should note that not all waves are dispersive. For example, sound has a velocity,

$$a = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2},$$

where p_0 and ρ_0 are the mean pressure and density respectively. γ is known as the adiabatic index (a ratio of the specific heats), which takes the value of around 1.4 in air. Sound waves are longitudinal (compressive) waves. You should note that there is no k dependence in their velocity. That is, all sound waves travel at the same speed, irrespective of wavelength. Another example of a non-dispersive system is that of shallow-water waves. Of course, in these non-dispersive systems phase and group velocities are essentially the same thing.

20.2 Group Velocity

In this section we will try to develop a more intuitive feel for what the group velocity in a dispersive system is. Suppose that we have a system that supports wave propagation. We will assume the dispersion relationship takes the general form $\omega = \omega(k)$. We know that in such a system the group velocity will depend on k. We can state the following properties about the group velocity.

Property - The group velocity is the speed at which an isolated wavepacket travels as a whole. In other words, the total wave energy propagates at the group velocity.

Proof - We will start by considering the motion of a wavepacket. We will represent this wavepacket by a Complex Fourier Integral (with the understanding that we are only taking the real part),

$$\eta(x,t) = \int_{-\infty}^{\infty} a(k) e^{i(kx-\omega t)} dk.$$

If η takes the form of a single wavepacket with almost constant wavenumber k_0, and if the amplitude varies slowly with x (i.e. the packet has a large number of wave crests), then plotting the amplitude against wavenumber we will get the following, figure 20.2. This,

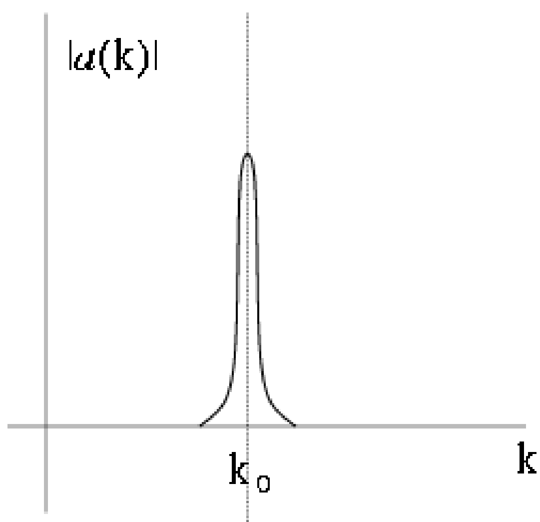

Figure 20.2: Amplitude-Wavenumber Plot

is almost like a delta function at k_0. If you are familiar with spectral analysis this is not surprising. All this graph is doing is showing the dominant modes in our system. In our specific case, these modes are waves of wavenumber near or equal to k_0. We should note that waves with wavenumber k far from k_0 have very small $|a(k)|$. Close to k_0 we can use a Taylor expansion to represent our dispersion relation.

$$\omega(k) = \omega(k_0) + (k - k_0)c_g + \text{h.o.t},$$

where c_g is the group velocity,

$$c_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}.$$

Now, we will substitute this expansion for $\omega(k)$ into the original Fourier Integral, (the idea being that for values of k far from k_0, where the expansion fails, $|a(k)|$ will be so small that it won't matter). After re-arranging some our terms we find,

$$\eta(x,t) = \underbrace{e^{i(k_0 x - \omega(k_0)t)}}_{(A)} \underbrace{\int_{-\infty}^{\infty} a(k) e^{i(k-k_0)(x-c_g t)} dk}_{(B)}$$

We see that our wavepacket η as been split into two parts. Part (A) represents a pure harmonic wave with wavenumber k_0. Part (B) represents a slowly modulating envelope this is a function of x and t only. We can see this through the particular combination of $x - c_g t$. That is, the envelope (packet) moves at the group velocity c_g.