Lecture 21

Deep Water Waves

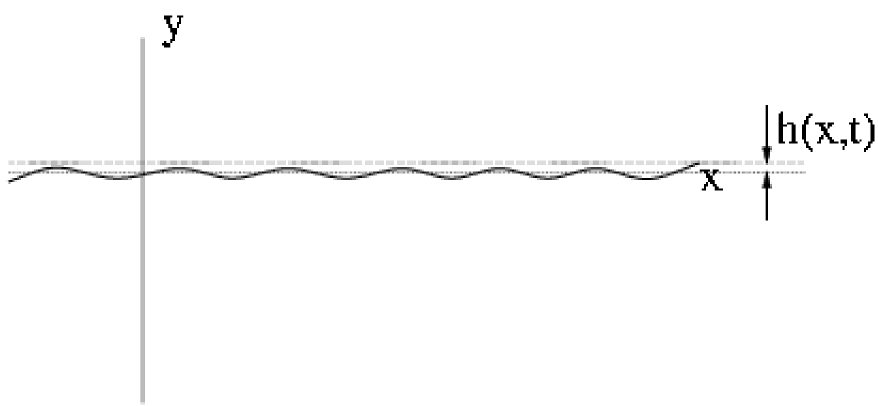

Figure 21.1: Deep Water Waves

Lets us investigate 2D water waves. In particular, we will consider the case of surfaces waves in deep water, figure 21.1.

We start by assuming we have an irrotational flow. This eliminates vorticity from our problem,

$$\partial_x v - \partial_y u = 0.$$

This may seem a drastic approximation, however, if the fluid was initially at rest (i.e. with zero vorticity), then by the conservational properties of the 2D vorticity equation, the flow will remain irrotational for all time. (This is only true if viscosity is unimportant). This allows us to represent our equations of motion in terms of a velocity potential.

$$\mathbf{u} = \nabla \phi.$$

Our incompressibility condition will, therefore, take the form of Laplace's equation,

$$\nabla^2 \phi = 0.$$

21.1 Boundary Conditions of the Free Surface

Now we must consider the all important boundary conditions. Our water waves will disturb the free surface from its mean (flat) position. We will denote this vertical distance, from the mean position, by $h(x,t)$. In other words,

$$y = h(x,t),$$

is the general equation for the free surface.

BC 1: The pressure p at the free surface is the atmospheric pressure p_0,

$$p = p_0 \text{ at } y = h(x,t),$$

where p_0 is a constant.

Therefore, we can use a Bernoulli equation to relate our velocity potential to the pressure via the momentum equation. Consider, the 2D incompressible Euler equation. In previous lectures, we have seen that this equation can be written in the following form,

$$\partial_t \mathbf{u} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla \left(\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2 + \lambda \right),$$

where $\lambda = gy$. Now, in our case we are considering an irrotational flow so the second term on the LHS disappears. Substituting the velocity potential ϕ we find,

$$\partial_t \nabla \phi = -\nabla \left(\frac{p}{\rho} + \frac{(\nabla\phi)^2}{2} + gy \right).$$

Now, integration gives,

$$\partial_t \phi + \frac{p}{\rho} + \frac{(\nabla\phi)^2}{2} + gy = c(t),$$

where $c(t)$ is an arbitrary function of time t. This is the Bernoulli's equation for an unsteady irrotational flow. Now applying our pressure condition at the free surface $y = h(x,t)$.

$$\partial_t \phi + \frac{p_0}{\rho} + \frac{(\nabla\phi)^2}{2} + gh = c(t).$$

Now, by choosing $c(t)$ to be p_0/ρ we can simplify this further and find that the pressure condition at the free surface can be written just in terms of ϕ. That is,

$$\partial_t \phi + \frac{(\nabla\phi)^2}{2} + gh = 0,$$

at $y = h(x,t)$.

BC 2: The Kinematic BC - This boundary condition basically stems from the idea that particles on the surface must remain at the surface. Imagine that all the surface particles are marked by dye at some instant. Now we will define the function $f(x,y,t)$ to be,

$$f(x,y,t) = y - h(x,t).$$

We may then claim that $f(x,y,t)$ is constant for any of the surface particles. (In fact, this constant is zero). Expressing mathematically such a conservation along fluid trajectories, we have

$$D_t f = \partial_t f + (\mathbf{u} \cdot \nabla) f = 0 \text{ at } y = h(x,t).$$

Now, we can re-write this in terms of h via the relationships,

$$\begin{aligned} \partial_t f &= -\partial_t h, \\ u\partial_x f &= -u\partial_x h, \\ v\partial_y f &= v. \end{aligned}$$

This gives us,

$$\partial_t h = v - u\partial_x h \text{ at } y = h(x,t). \tag{21.1}$$

Intuitively, this above argument may not seem very obvious! So, we will consider two special cases to check that the relationship is correct, and then consider a more physical interpretation of equation (21.1). Firstly, if the free surface stays horizontal, then the second term on the RHS of equation (21.1) is zero and we find the height of the free surface h changes with the vertical velocity v, which is correct. Secondly, if the free surface is stationary, $h = h(x)$ then $v/u = dh/dx$. This implies that the slope of the streamlines at $y = h(x)$ is equal to to the slope of the free surface, as it should be in this particular case.

Okay, now for a more intuitive understanding of equation (21.1). We can separate the RHS into two parts. The first term is a vertical component, this represents how h varies due to the vertical velocity. The second term represents how h varies due horizontal effects, that is, how it changes as it gets advected along.

21.2 Small Amplitude Waves

To find small amplitude wave solutions to our system, we need to linearise our equations and boundary conditions. From our pressure condition we get,

$$\partial_t \phi + gh = 0 \text{ at } y = 0, \tag{21.2}$$

where we ignore the second order terms. While our kinematic condition becomes,

$$\partial_t h = v \text{ at } y = 0, \tag{21.3}$$

where we have ignored the $u\partial_x h$ term. We can ignore this term because if $v \sim u$ then $\partial_t h = v$ when $\partial_x h$ is small, and this is true when h is small. This analysis is for deep water waves, that is when $h \ll$ the wavelength.

21.3 Dispersion Relation

We will now seek a sinusoidal travelling wave solution. Therefore, we will let,

$$h = A\cos(kx - \omega t),$$

where A is the amplitude of the surface displacement, ω is the frequency and k is the wavenumber. Looking at the equations (21.2) and (21.3) we see that the velocity potential must have the form,

$$\phi = f(y)\sin(kx - \omega t).$$

Plugging this into the continuity equation we find,

$$-fk^2 \sin(kx - \omega t) + f'' \sin(kx - \omega t) = 0.$$

Therefore,

$$f'' - k^2 f = 0.$$

This has a solution of the form,

$$f = Ce^{ky} + De^{-ky}.$$

If we take $k > 0$, then if the water is of infinite depth, we must set $D = 0$. Likewise, if $k < 0$ we must set $C = 0$ to ensure that the velocity is bounded as $y \to -\infty$. Without loss of generality we will assume that $k > 0$, in which case we find,

$$\phi = Ce^{ky} \sin(kx - \omega t).$$

Substituting this equation and (21.3) into the linearised free surface conditions (21.2) and (21.3), we obtain,

$$Ck = A\omega,$$
$$-C\omega + gA = 0,$$

Therefore, we find,

$$\phi = \frac{A\omega}{k} e^{ky} sin(kx - \omega t),$$

and that,

$$\omega^2 = gk.$$

This latter equation is the dispersion relation. It immediately tells the phase and group velocities of our deep water waves are,

$$c_{ph} = \frac{\omega}{k} = \left(\frac{g}{k}\right)^{1/2},$$
$$c_g = \partial_k \omega = \frac{1}{2} c_{ph}.$$