Lecture 22

Stability of a Plane Shear Flow

In this lecture we will consider the phenomenon of flow instability. The quickest way to get an example of instability is to generalise the deep-water wave analysis to the case where the upper and the lower fluids (air and water in the previous lecture) have arbitrary densities ρ_2 and ρ_1, (as formulated on one of the example sheets). The resulting dispersion relation is,

$$\omega^2 = gk\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}.$$

One can see that if the upper fluid is heavier than the lower, $\rho_2 > \rho_1$, then the ω is purely imaginary and any initial perturbations grow exponentially.

22.1 The Inviscid Theory

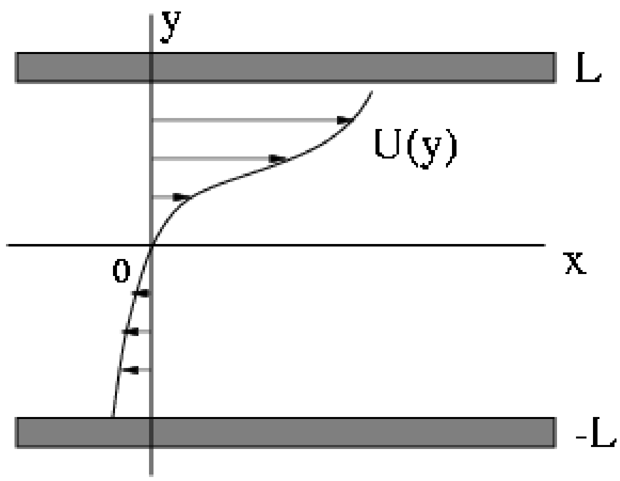

Figure 22.1: Plane Shear Flow between two flat plates

Let us now consider the stability of a plane shear flow. We will consider an inviscid flow between two flat plates, in 2D. The plates are positioned at $y = -L$ and $y = L$, see figure 22.1. Our basic equations are the incompressible Euler equations. In components we write,

$$\partial_t u + u\partial_x u + v\partial_y u = -\frac{1}{\rho}\partial_x p,$$
$$\partial_t v + u\partial_x v + v\partial_y v = -\frac{1}{\rho}\partial_y p,$$
$$\partial_x u + \partial_y v = 0.$$

The parallel shear flow has the velocity profile,

$$\mathbf{u}_0 = [U(y), 0, 0],$$

and this is an exact solution to the Euler equations for any $U(y)$, where the pressure is a constant p_0.

As is the norm, in any linear stability analysis, we will now consider small perturbations to our parallel shear flow. In our case these small disturbances will be 2D. Therefore, we let,

$$\mathbf{u} = [U(y) + u_1, v_1, 0],$$

where u_1 and v_1 are small functions of x,y and t. We also have to take into account that pressure will also change and denote this perturbation by p_1. Substituting our perturbations into our governing equations and linearising we find,

$$\partial_t u_1 + U\partial_x u_1 + v_1\partial_y U = -\frac{1}{\rho}\partial_x p_1,$$
$$\partial_t v_1 + U\partial_x v_1 = -\frac{1}{\rho}\partial_y p_1,$$
$$\partial_x u_1 + \partial_y v_1 = 0.$$

One should note that $\partial_x U = 0$, so this term is not present in the momentum equation. These equations only have coefficients that depend on y alone. Therefore, we will examine the stability of our system with wave solutions of the form,

$$u_1 = Re\left[\hat{u}(y)e^{i(kx-\omega t)}\right],$$
$$v_1 = Re\left[\hat{v}(y)e^{i(kx-\omega t)}\right],$$
$$p_1 = Re\left[\hat{p}(y)e^{i(kx-\omega t)}\right].$$

Substituting these into our linearised equations and cancelling the exponential terms we find,

$$-i(\omega - Uk)\hat{u} + \hat{v}\partial_y U = -\frac{1}{\rho}ik\hat{p}, \tag{22.1}$$
$$-i(\omega - Uk)\hat{v} = -\frac{1}{\rho}\partial_y \hat{p}, \tag{22.2}$$
$$ik\hat{u} + \partial_y \hat{v} = 0. \tag{22.3}$$

Differentiating equation (22.1) and substituting it into equation (22.2), and using equation (22.3) for $\hat{u}$, we find the following differential equation for the amplitude of our perturbation

$$\hat{v}'' + \left(\frac{kU''}{\omega - Uk} - k^2\right)\hat{v} = 0,$$

where the primes denote differentiation with respect to y. The boundary conditions to this equation are,

$$\hat{v}(L) = \hat{v}(-L) = 0.$$

This can be regarded as an eigenvalue problem for ω. Now you might think we have come as far as we can. That is, we can only solve this equation by specifying a velocity profile $U(y)$ and then solving the corresponding eigenvalue problem numerically. However, *Lord Rayleigh* (1880) devised a clever argument which allows us to obtain a condition for stability.

We start by multiplying our differential equation by $\hat{v}^*$ the complex conjugate of $\hat{v}$ and then integrating between $-L$ and L. This gives,

$$\int_{-L}^{L} \hat{v}^* \hat{v}'' dy + \int_{-L}^{L} \left(\frac{kU''}{\omega - Uk} - k^2 \right) |\hat{v}|^2 dy = 0.$$

This is useful because we can obviously say something about $|\hat{v}|^2$, (it is greater than or equal to zero), without solving for $\hat{v}$. Integrating by parts we find,

$$[\hat{v}' \hat{v}^*]_{-L}^{L} - \int_{-L}^{L} |\hat{v}|^2 dy + \int_{-L}^{L} \left(\frac{kU''}{\omega - Uk} - k^2 \right) |\hat{v}|^2 = 0.$$

The first term vanishes because of our boundary condition. We now separate our waves frequency into real and imaginary parts $\omega = \omega_R + i\omega_I$ which gives,

$$-\int_{-L}^{L} |\hat{v}|^2 dy + \int_{-L}^{L} \left(\frac{(\omega_R - Uk - i\omega_I)kU''}{|\omega - Uk|^2} - k^2 \right) |\hat{v}|^2 = 0.$$

Now the real and imaginary parts of this equation must be both separately zero. The imaginary parts tell us,

$$\omega_I k \int_{-L}^{L} \frac{U''|\hat{v}^2|}{|\omega - Uk|^2} dy = 0.$$

Now suppose there is at least one mode which has $\omega_I > 0$, this corresponds to an exponential growth of the amplitude in time. According to the above equation this is impossible unless $U''(y)$ changes sign somewhere on the interval $(-L, L)$. Otherwise the integral cannot vanish. This result gives rise to *Rayleigh's Inflection Point Theorem.*

Rayleigh's Inflection Point Theorem - A necessary condition for the linear instability of an inviscid shear flow $U(y)$ is that $U''(y)$ should change sign somewhere in the flow.

Example 1 - A Poiseuille flow, $U = (L^2 - y^2)U_{MAX}/L^2$ is stable because it doesn't have an inflection point.

Example 2 - A Tangential Discontinuity is unstable because there is an inflection point at the origin, see figure 22.2.

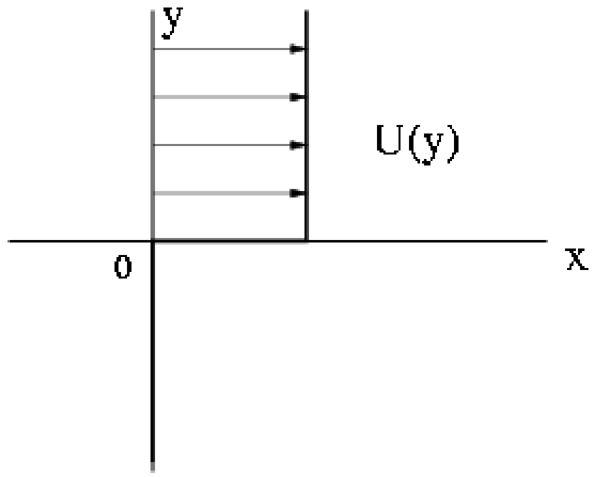

Figure 22.2: Tangential Discontinuity

22.2 The Viscous Theory

Repeating the earlier analysis but now with a viscous fluid we would obtain the following,

$$i\nu(\hat{\psi}'''' + 2k^2\hat{\psi}'' + k^4\hat{\psi}) + (Uk - \omega)(\hat{\psi}'' - k^2\psi) - U''k\hat{\psi} = 0,$$

where our perturbations u_1 and v_1 have been written in terms of the streamfunction,

$$u_1 = \partial_y \psi,$$
$$v_1 = -\partial_x \psi,$$

and

$$\psi = Re\left[\hat{\psi}(y)e^{i(kx-\omega t)}\right].$$

In general this can only be solved numerically. However, in the case of a Poiseuille flow one can use an analytic approach. One should note that although viscosity can, in some cases, act as a stabilizing mechanism, it can also introduce instabilities into a flow which, according to inviscid theory, is apparently stable.