Lecture 23

Turbulence

Turbulence is perhaps the most complicated subject in Fluid Dynamics, and the theory of turbulence is still very far from being complete. The study of turbulence began as early as 1500 when Leonardo sketched his wonderful drawings depicting turbulence in different situations. It is at this time that he made the following very important classifications of turbulent states:

- **1.** Forced Turbulence,
- **2.** Decaying Turbulence,
- **3.** Self-substained Turbulence.

These classifications, especially points (1) and (2), distinguish between the principally different cases found in modern turbulence theory. So what is turbulence? Generally, by turbulence we mean a chaotic motion of fluid elements involving eddies of many different sizes. These eddies non-linearly interact, creating and transferring their energy to new eddies. The fluid motion is described by the NSE's, which are deterministic. However, we cannot exactly predict, or reproduce, the instantaneous velocity profiles obtained in a single turbulent experiment. This is because the system is very sensitive to any, even very small, changes in its initial conditions. However, it is believed that one can describe turbulence statistically. That is, some averaged characteristics of the turbulent velocity field are predictable and reproducible. The statistical theory of turbulence is very complicated and is far from being complete. However, one can understand several important facts and concepts about turbulnce at a very elementary level.

23.1 3D Turbulence

We often use statistical methods to determine properties of a turbulent system. In such a case it makes little sense to talk about the average of say the velocity field $< u(\mathbf{x}, t) >$ since we can choose a reference frame where this is zero. Instead we look at the averaged value of the square of the velocity field, (Energy density),

$$E = \frac{1}{2} < u^2 >,$$

or correlations, for example

$$< u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{r}, t) > .$$

Consider a turbulent system. Let us suppose that the energy, is pumped into eddies of size L. These eddies will interact and distort each other, creating smaller eddies. In turn, these smaller eddies will break into even smaller eddies, and so on, until a very small scale is reached where the eddies are dissipated by viscosity. Thus, the energy will cascade over the scales until it reaches the scale of viscous dissipation (where it will be converted into heat), figure 23.1. This picture of the energy cascade over the turbulent scales was

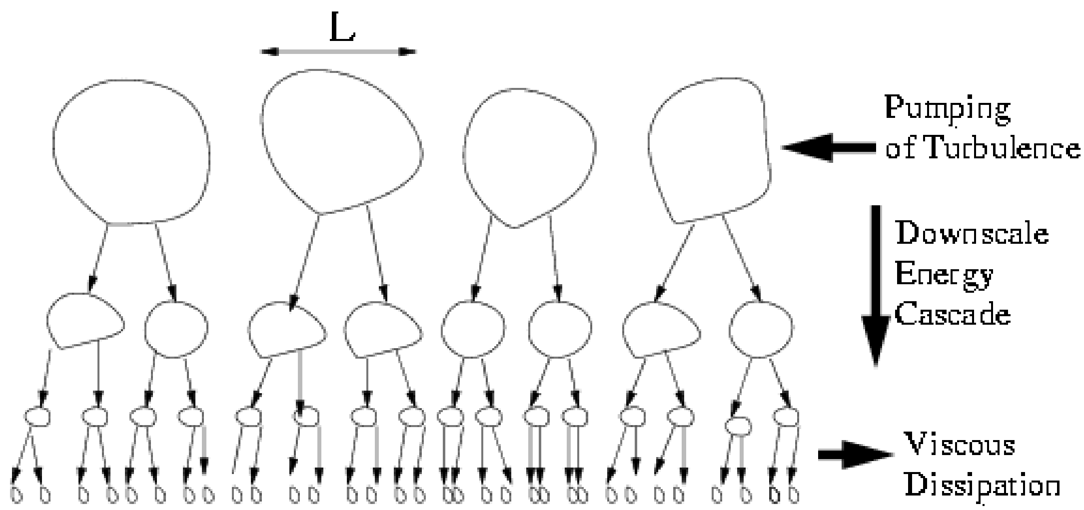

Figure 23.1: Energy Cascade

introduces by Richardson in 1922. Richardson, in turn, was inspired by Jonathan Swift's verse:

"So, nat'ralists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller yet to bite 'em,
And so proceed ad infinitum."

We should note that Richardson's cascade is *local.* That is, the energy, which is trying to reach the very small-scales, cannot bypass the intermediate scales by *"leaping"* over them.

The concept of Richardson's energy cascade plays a central role in the theory of turbulence. For example, it allows us to find energy distribution over the scales using a simple dimensional argument. However, we should ask, mathematically speaking, what this scale actually is? The answer to this question can be found by using *Fourier Transforms.* We should first define what we mean by *Spectral Energy Density.*

Definition The Spectral Energy Density in d-dimensional k-space is,

$$\tilde{E}(\mathbf{k}) = \frac{1}{2} \int < u(\mathbf{x}) u(\mathbf{x} + \mathbf{r}) > e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}.$$

We can show that,

$$E(\mathbf{x}) = \frac{1}{2} < u^2 >= \int \tilde{E}(\mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^d} = \int \hat{E}(k) dk,$$

where d is the dimension of our system. $\hat{E}(k)$ is the 1-dimensional k-space spectral energy density. To link $\hat{E}(k)$ to $\tilde{E}(\mathbf{k})$ we notice in 3D that,

$$d\mathbf{k} = 4\pi k^2 dk.$$

Therefore, we see that,

$$\hat{E}(k) = \frac{2k^2}{(2\pi)^2} \tilde{E}(\mathbf{k}) = \frac{k^2}{(2\pi)^2} \int < u(\mathbf{x}) u(\mathbf{x}+\mathbf{r}) > e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}.$$

As an aside, in 2D we would have,

$$d\mathbf{k} = 2\pi k dk,$$

which implies,

$$\hat{E}(k) = \frac{k}{(2\pi)} \tilde{E}(\mathbf{k}) = \frac{k}{4\pi} \int < u(\mathbf{x}) u(\mathbf{x}+\mathbf{r}) > e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}.$$

Note that in statistically homogeneous turbulence $< u(\mathbf{x}) u(\mathbf{x}+\mathbf{r}) >=< u(0) u(\mathbf{r}) >$ since it is only the relative position of measurements that are important. In terms of k the turbulent length scale l is $2\pi/k$, and the spectral energy density $\hat{E}(k)$ plays the role of the energy distribution over the scales.

Now, following Kolmogorov's 1941 paper, we can use a dimensional argument to derive the energy spectrum $\hat{E}(k)$. Let us suppose that we pump our turbulent energy into a wavenumber $k = k_f$ (large length scale) which is much less than the wavenumber k_v (small scale) where the dissipation becomes important, figure 23.2. The region $k_f < k < k_v$,

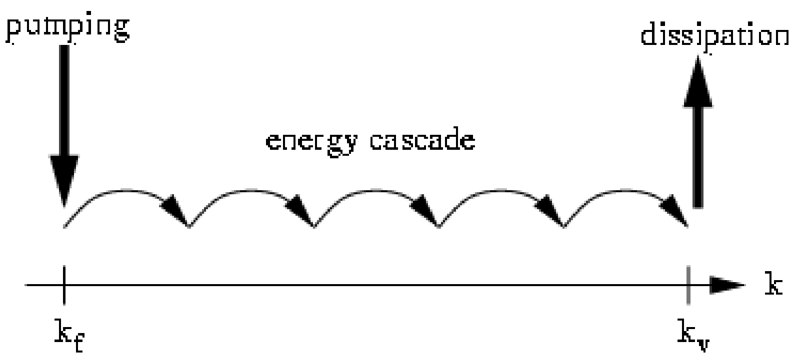

Figure 23.2: Energy Cascade with Pumping and Dissipation

where there is no pumping or dissipation, is called the *inertial range*.

Kolomorgov's Universality Hypothesis: For $k_f \ll k \ll k_v$, within the inertial region, the properties of turbulence depends only on the energy cascade rate (flux), but not on the specific way the energy is produced at k_f and dissipated at $k > k_v$.

The energy cascade rate in stationary turbulence is equal to the rate ϵ at which the energy is dissipated at $k > k_v$. Thus, for $k_f \ll k \ll k_v$ there are only two dimensional quantities available in the problem: ϵ and k. We have to arrange them in such a way that we obtain the dimension of $\hat{E}(k)$. The dimension of k is obviously $1/l$. The dimension of ϵ is,

$$\left[\frac{\mathbf{u}^2}{t}\right] = \frac{l^2}{t^3}.$$

Now the dimension of $\hat{E}$ is,

$$\left[\frac{\mathbf{u}^2}{k}\right] = \frac{l^3}{t^2}.$$

There is *only one* combination of ϵ and k which gives the same dimension as $\hat{E}$, that is,

$$\hat{E} = C\epsilon^{2/3}k^{-5/3}.$$

This is the famous *Kolmogorov Spectrum.* C is a constant of order one called the Kolmogorov constant.

Inspite of the simplicity of this derivation, the Kolomorgorov Spectrum is, by far, the most central of all results in turbulence theory. It is observed experimentally in the atmosphere, the oceans and under laboratory conditions. However, all attempts to derive this spectrum rigorously from NSE's (using a statistical theory) have so far been unsuccessful.