Lecture 24

2D Turbulence

Consider a 2D fluid such as, for example, the large-scale motions of an atmosphere or ocean. In this 2D case there are two integrals of motion:

$$E_{tot} = \frac{1}{2}\int \mathbf{u}^2 d\mathbf{x} \rightarrow \text{ total energy}$$
$$P = \frac{1}{2}\int \omega^2 d\mathbf{x} = \frac{1}{2}\int (\nabla \times \mathbf{u})^2 d\mathbf{x} \rightarrow \text{ enstrophy}$$

Naturally, in homogeneous turbulence E_{tot} and P are infinite and we must deal with the densities of these quantities (these being finite). We already know that in terms of the energy spectrum $\hat{E}(k)$,

$$\frac{Energy}{UnitArea} = \frac{1}{2} < \mathbf{u}^2 >= \frac{1}{2}\int \hat{E}(k)dk.$$

Now, using $\omega_k = i\mathbf{k} \times \mathbf{u}_k$ we also have:

$$\frac{Enstrophy}{UnitArea} = \frac{1}{2} < \omega^2 >= \frac{1}{2}\int k^2 \hat{E}(k)dk,$$

where ω_k and $\mathbf{u_k}$ are the fourier transform of the vorticity and velocity. If we force turbulence at some scale where the viscosity is unimportant, then both the energy and enstrophy will cascade over the turbulent scales. We will now show that these two cascades happen in opposite directions to each other, in wavenumber space.

Suppose that we are forcing turbulence at some k_f, and that there is a viscous dissipation in the system for $k > k_v \gg k_f$. Suppose also that there is some kind of dissipation for the large-scales $k < k_L \ll k_f$ due to, for example, friction induced by the surface boundary layer, figure 24.1. We now assume some finite amount of energy and enstrophy is produced in unit time at k_f, (we will set $E \sim P \sim 1$). Suppose that some fraction of energy, comparable to the energy produced at k_f, is dissipated at $k > k_f$. However, the spectral density of enstrophy, $k^2|\mathbf{u_k}|^2$, is $\mathbf{k}^2$ greater than the spectral density of the energy $|\mathbf{u_k}|^2$. Therefore, there will be $\sim (k_v/k_f)^2$ times more enstrophy dissipated at $k > k_v$ than the amount of enstrophy produced at $k \sim k_f$. In a stationary state this is impossible, because the amount of enstrophy created at the source must be equal to the amount dissipated at the sink. Therefore, the energy can only be dissipated in the $k < k_L$ region.

Now, suppose that a finite amount of enstrophy is dissipated at $k < k_L$. But in this range the spectral density of energy is k^{-2} times greater than the spectral density of the

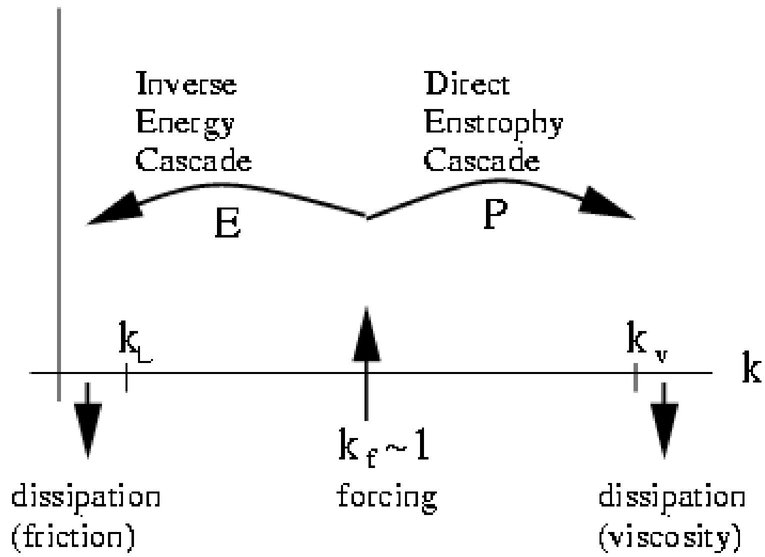

Figure 24.1: 2D Turbulence Inverse Energy Cascade

enstrophy because $k \ll 1$. Therefore, there will be more energy dissipated (by a factor of $(k_f/k_L)^2$) at $k < k_L$ than the amount produced at $k \sim k_f$. Again, this is impossible in stationary turbulence. Thus, the enstrophy can only dissipate at $k > k_v$, i.e. in the viscous range.

Summary - The energy cascade is inverse (with respect to the 3D turbulence case), whereas the enstrophy cascade is direct. *George Batchelor* was the first to use the above argument in explaining the different directions of these two cascades.

As was stated in the last lecture, the range $k_L < k < k_f$ is called the inertial interval. Flowing the logic of Kolmogorov's universality hypothesis, we can assume that in the range $k_L < k < k_f$ the energy spectrum is determined by the energy dissipation rate ϵ only. Whereas, for $k_f < k < k_v$ the energy spectrum will only be determined by the enstrophy dissipation rate η. Therefore, we can use a dimensional argument again to find the energy spectrum $\hat{E}(k)$ in both these ranges. This was done by *Robert Kraichnan* in the early 60's. Actually, for the energy cascade range the argument is identical to the one given in the last lecture. Therefore, for $k_L < k < k_f$ we have,

$$\hat{E} = C_1 \epsilon^{2/3} k^{-5/3},$$

where C_1 is a constant of order one (but different from the Kolomogorov constant in the 3D case!). Now, for the other range. The dimension of η is,

$$[\eta] = \left[\frac{\omega^2}{t}\right] = \frac{1}{t^3}.$$

The only combination of η and k that gives the correct dimensions for $\hat{E}(k)$ is,

$$\eta^{2/3} k^{-3}.$$

Therefore, for $k_f < k < k_v$ we have,

$$\hat{E} = C_2 \eta^{2/3} k^{-3}.$$

This gives us the famous Kraichnan's spectrum, figure 24.2. In numerical and laboratory

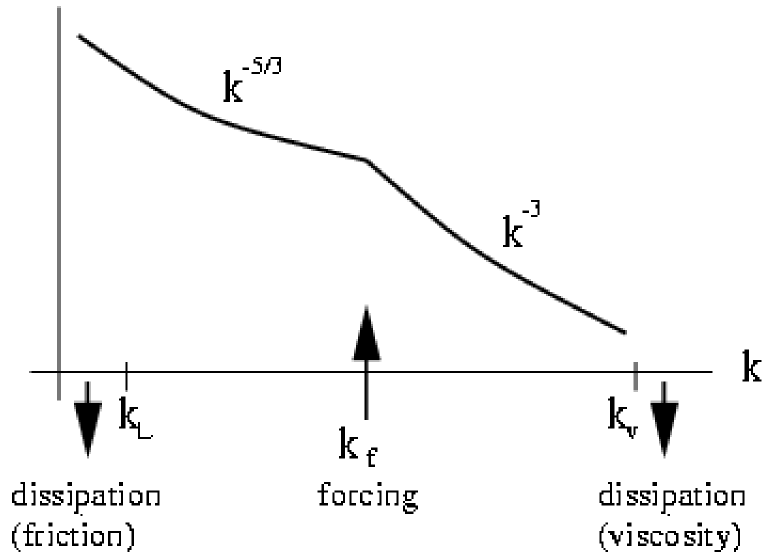

Figure 24.2: 2D Turbulence - Kraichnan's Spectrum

experiments the k^{-3} spectrum is indeed observed. However, the $k^{-5/3}$ is hardly ever observed in 2D turbulence. The reason for this is that the dissipation at $k < k_L$ is insufficient (in real situations) to dissipate the energy produced at k_f. Instead, the energy tends to "*pile-up*" at the largest scale of the system (i.e smallest k) because of the inverse energy cascade.

24.1 2D Vortex Dynamics and Vortex Pairs

Understanding of 2D turbulence is often helped by considering simple 2D vortex solutions, which are the subject of this section. (See also the handout on vortex pairs). In particular, we will show in the end of this lecture, how the inverse energy cascade and direct enstrophy cascade can be interpreted in terms of the elementary vortex dynamics.

24.1.1 Vortex Pairs

We should really ask what happens to vortices of the same sign. Consider two blobs of vorticity, where,

$$\mathbf{\Omega} = \begin{cases} 1 \text{ inside vortices} \\ 0 \text{ outside} \end{cases} .$$

This problem was investigated theoretically by Melander and Zabusky in the 80's. We find the two blobs merge and slowly wrap around each other, figure 24.3. A similar problems, considered by Onsager in 1949, was that of a sea or gas of point vortices (of mixed sign). In this case one finds that vortices of the same sign agglutinate.

Figure 24.3: Vortex Merger

The inverse cascade of energy and cascade of enstrophy in 2D turbulence can be understood in the context of 2D vortex dynamics. In particular two vortices which merge will produce a larger vortex (i.e transfering their energies to the large-scales). While the small-scale filaments produced in such a process carry away the enstrophy.