# Lecture 25

# Turbulent Boundary Layer

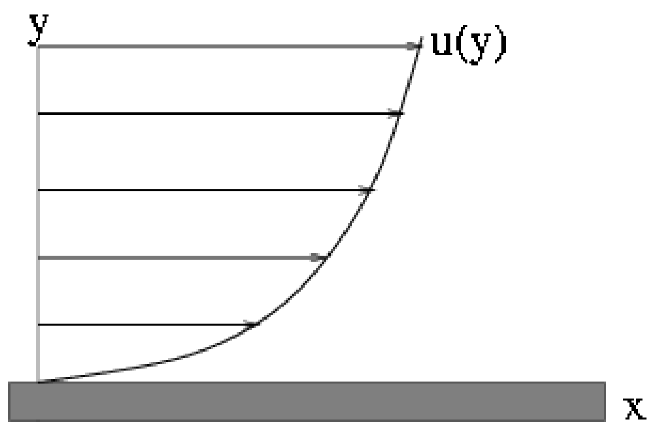


Figure 25.1: Mean Turbulent Boundary Layer Velocity Profile

Under high Reynold's numbers any boundary layer becomes unstable. Our linear instability analysis will fail to be valid when the unstable disturbances grow to high amplitudes. The full-scale nonlinear dynamics will be present then, with chaos and turbulence being one of its manifestations. Consider a BL over a flat plate, where the *mean* velocity profile looks something like that found in figure 25.1. Note that this is only the mean profile. An instantaneous velocity field will involve turbulent fluctuations, with velocities having a dependence on all three spatial coordinates and time, figure 25.2. Let us denote the frictional force, produced by the fluid on a unit area of the surface, by $\sigma$. We should note that $\sigma$ is the fluids flux of $x$-momentum in the negative $y$-direction, (just like the laminar BL case we considered earlier). The flux of $x$-momentum $\sigma$ is constant for any $y$ in stationary turbulence because the momentum is a conserved quantity in fluids. The momentum flux originates from the velocity gradient,

$$\frac{\partial u}{\partial y}. \tag{25.1}$$

Indeed, if all parts of the fluid moved with the same velocity then there wouldn't be a momentum flux. Now, using dimensional arguments, we notice that the only combination of $\sigma$ (which has the dimensions of $u^2 = l^2/t^2$ if $\rho = 1$) and $y$ having the same dimensions as the velocity gradient (25.1) is,

$$\frac{\sigma^{1/2}}{y}.$$

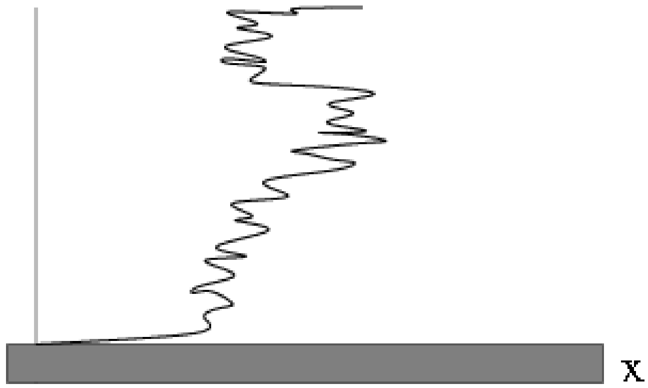

Figure 25.2: Instantaneous Velocity Profile

Therefore, we find,

$$\frac{\partial u}{\partial y} = \frac{\sigma^{1/2}}{\aleph y}, \tag{25.2}$$

where $\aleph$ is a constant of order 1, known as *Karman's* constant.

Numerical values, obtained from experiment, place $\aleph$ at approximately 0.4. Solving equation (25.2) gives,

$$u = \frac{\sigma^{1/2}}{\aleph}(\ln y + C).$$

This law was first obtained independently by Von Karman and Prandtl. Landau found the constant $C$ via the BC's. Firstly, we should note that there is a thin laminar layer between the turbulent BL and the wall. The characteristic velocity in turbulent pulsations is $\sigma^{1/2}$, while the viscosity starts to become important at $y = y_0$ when,

$$Re = \frac{y_0 \sigma^{1/2}}{\nu} \sim 1.$$

Or, rearranging we find,

$$y_0 = \frac{\nu}{\sigma^{1/2}}.$$

For the laminar BL, we have already obtained,

$$\sigma = \nu \frac{du}{dy}.$$

From this we find,

$$u = \frac{\sigma y}{\nu}.$$

This layer is called the *viscous sub-layer*, figure 25.3. By matching $u$ with the characteristic turbulence velocity $\sigma^{1/2}$ at $y = y_0$, we find the constant $C$ to be,

$$C = -\ln y_0.$$

Thus finally, we arrive at the following expression for the mean velocity in the turbulent BL,

$$u = \frac{\sigma^{1/2}}{\aleph} \ln\left(\frac{y\sigma^{1/2}}{0.13\nu}\right)$$

This distribution is called the *logarithmic BL*. It is observed experimentally in turbulent pipe flows and found in many other situations, both in the laboratory and in nature.

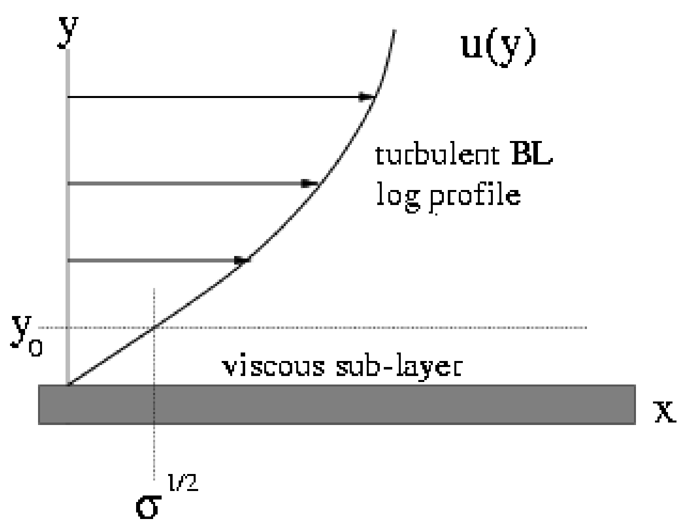


Figure 25.3: Viscous Sub-Layer