Lecture 26

Weak Turbulence

We will now move on to consider weak turbulence in the context of waves. In a linear system it is a well known fact that waves don't interact. They past through each other, without change in energy, frequency or wavenumber. Solutions are made up of sinusoidal waves, for example

$$h(x,t) = \sum A(k)e^{ikx-i\omega(k)t} + c.c,$$

where $c.c$ is the complex conjugate. However, it is more natural to consider the case where waves do interact. In weak turbulence we assume that that these interactions are small, that is, that any non-linearity in our modified wave equation is small. The advantage of this is that, the first approximation, our system is still made up of distinct sinusoidal type solutions. However, these dispersive waves will now interact weakly. From previous discussions of waves we derived various dispersion relations for different examples. For instance, for deep water waves we found,

$$\omega = \sqrt{gk},$$

while for capillary waves we found,

$$\omega = \sigma^{1/2}k^{3/2},$$

and for sound waves we observed that,

$$\omega = c_s|k| \text{ with } c_s^2 = \frac{\partial p}{\partial \rho}.$$

Interactions between waves are characterised by different resonance conditions. For a three wave interaction we have,

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2,$$
$$\omega(\mathbf{k}) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2).$$

If these conditions can be solved simultaneously then we have a three wave process. However, if these conditions are not satisfied then one must consider the possibility of a four wave interaction, where

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4,$$
$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) + \omega(\mathbf{k}_4).$$

If no such relation (where $n = 3{,}4, \cdots, \infty$) is satisfied the system is integrable.

Whether a given process occurs is determined by the dispersion relationship and the non-linearity. Consider the general relationship,

$$\omega = \lambda k^{\alpha},$$

where λ and α are constants. Then, if $\alpha > 1$ the 3 wave conditions have solutions (decaying dispersion relations), while if $\alpha < 1$ the 3 wave solutions are not satisfied. Trivial solutions are, of course, always possible. So why is the value of α so significant? It is easier to understand if we consider sketches of the relevant dispersion curves. Figure 26.1 shows a dispersion relationship for $\alpha > 1$, here we see the resonance conditions can easily be satisfied. Figure 26.2 shows the opposite case where $\alpha < 1$, here we see the nature of the curves make it impossible for us to satisfy the resonance conditions.

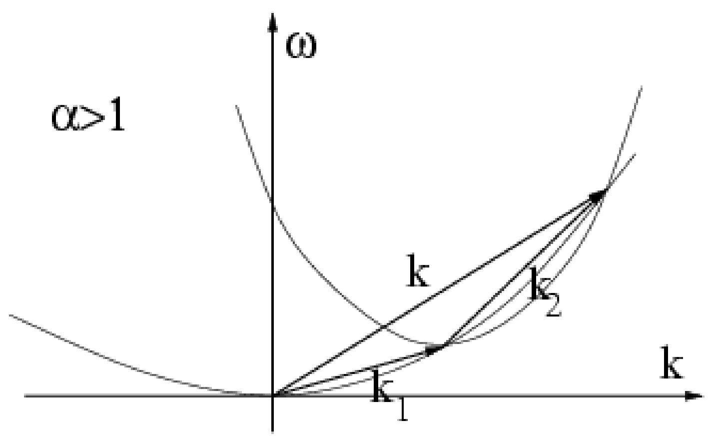

Figure 26.1: Dispersion Curves for $\alpha > 1$

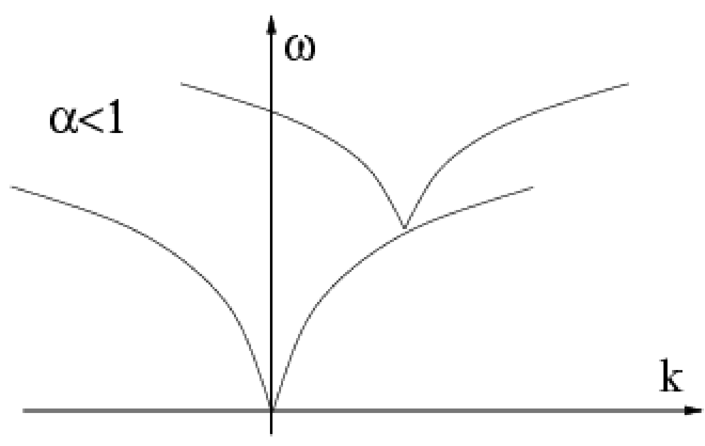

Figure 26.2: Dispersion Curves for $\alpha < 1$

Interactions result in the transfer of energy between different wavenumbers. We can defined the spectral energy via (c.f. lecture 23),

$$< \hat{u}(\mathbf{k})\hat{u}(\mathbf{k}') >= (2\pi)^d \tilde{E}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}').$$

Is this expression correct? We can check by direct substitution for the Fourier transforms of u.

$$
\begin{aligned}
< \hat{u}(\mathbf{k})\hat{u}(\mathbf{k}') > &=< \int u(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \int u(\mathbf{x}') e^{-i\mathbf{k}'\cdot\mathbf{x}'} d\mathbf{x}' >, \\
&= \int e^{-i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}'\cdot\mathbf{x}'} < u(\mathbf{x})u(\mathbf{x}') > d\mathbf{x}d\mathbf{x}', \\
&= \int e^{-i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}'\cdot\mathbf{x}'} < u(0)u(\mathbf{x}'-\mathbf{x}) > d\mathbf{x}d\mathbf{x}', \\
&= \int \underbrace{e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}}_{\Rightarrow (2\pi)^d \delta(\mathbf{k}+\mathbf{k}')} \underbrace{\int e^{-i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{x})} < u(0)u(\mathbf{x}'-\mathbf{x}) > d\mathbf{x}d\mathbf{x}'}_{\Rightarrow \tilde{E}(\mathbf{k}')}, \\
&= (2\pi)^d \tilde{E}(\mathbf{k}) \delta(\mathbf{k}+\mathbf{k}').
\end{aligned}
$$

Recall from lecture 23 we can relate $\tilde{E}(\mathbf{k})$ to $\hat{E}(k)$. Therefore, in 2D we find,

$$
< \hat{u}(\mathbf{k})\hat{u}(\mathbf{k}') > \frac{k}{(2\pi)^3} = \hat{E}(k)\delta(\mathbf{k}+\mathbf{k}').
$$

As an aside, in 3D we have

$$
< \hat{u}(\mathbf{k})\hat{u}(\mathbf{k}') > \frac{2k^2}{(2\pi)^5} = \hat{E}(k)\delta(\mathbf{k}+\mathbf{k}').
$$

The total energy per unit volume (Energy Density) is given by,

$$
\frac{\text{Total Energy}}{\text{Unit Volume}} =< u^2 >= \int \hat{E}(k) dk.
$$

(Note that in wave systems the *"mean potential energy = mean kinetic energy"*. This is why we don't have a factor of a 1/2 in front of the $< u^2 >$ as we did with the Navier-Stokes turbulence found in lecture 21). This will result in a turbulent spectrum, as in figure 26.3. This is similar to the Kolmogorov picture of Navier-Stokes turbulence we saw in lecture 21. One can imagine forcing waves at some wavenumber k_f and dissipating them at $k_d >> k_f$. In between k_f and k_d the energy cascades down scale without dissipation. The weak turbulence approach allows us to describe our system through *kinetic equations* which symbolically have the form,

$$
\partial_t \hat{E}_k = -\partial_k F + \text{ forcing } + \text{ dissipation},
$$

where F is the energy flux (due to the cascade) which is determined by a particular non-linear interaction process. For 3-wave processes we have the energy dissipation rate $\epsilon = \partial_t E \sim F \sim \hat{E}^2$ while for 4-wave we have $\epsilon \sim \hat{E}^3$.

We will finish off by trying to derive the energy spectra resulting from 3-wave and 4-wave processes using a dimensional arguments. One should note that we now have an extra dimensional parameter to take into account from the dispersion relationship, for example g for gravity waves, and σ for capillary waves. We also have the added constraint that energy dissipation $\epsilon \sim \hat{E}^2$ for the 3-wave case and $\epsilon \sim \hat{E}^3$ for the 4-wave case. We

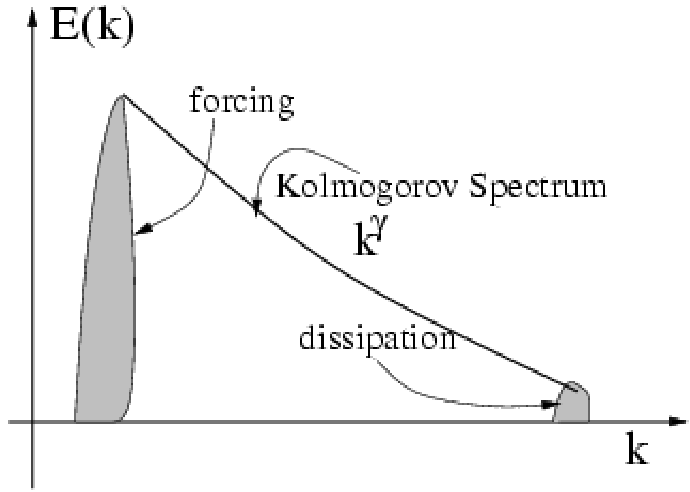

Figure 26.3: Energy Spectrum

now look for combinations of k,λ and ϵ such that our dimensional constions are satisfied. We have,

$$dim(\hat{E}) = \frac{[L]^4}{[T]^2},$$
$$dim(\epsilon) = \frac{[L]^3}{[T]^3},$$
$$dim(\lambda) = \frac{[L]^\alpha}{[T]},$$

giving for the 3-wave case,

$$\hat{E} = c\epsilon^{1/2}k^{(\alpha-5)/2},$$

and for 4-wave,

$$\hat{E} = c\epsilon^{1/3}\lambda k^{\alpha-3}.$$

Now because $\alpha = 1/2$ in the case of gravity waves, this must be a 4-wave process. While for capillary waves we have $\alpha = 2/3$ and $\lambda = \sqrt{\sigma}$ making it a 3 wave process. The corresponding spectra are therefore,

$$\hat{E}(k) = c\epsilon^{1/3}\sqrt{g}k^{-5/2} \text{ deep water waves,}$$
$$\hat{E}(k) = c\epsilon^{1/2}k^{-7/4} \text{ capillary waves.}$$