Lecture 27

Dynamics of Compressible Fluids

Throughout this course we have been mainly talking about incompressible fluids. However, some important and interesting phenomena occur in compressible fluid dynamics. For a compressible system, we have the following governing equations.

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p,$$
$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0.$$

Notice that the continuity equation now takes into account the variation of density.

27.1 1D Motions

Consider the 1D case,

$$\partial_t u + u\partial_x u = -\frac{1}{\rho}\partial_x p, \tag{27.1}$$
$$\partial_t \rho + \partial_x(\rho u) = 0. \tag{27.2}$$

Since ρ is now a variable we need another equation to close the system, usually this equation is given in terms of energy or entropy to link p with ρ. The material conservation of entropy can be written as,

$$D_t S = 0,$$

where $S = (p, \rho)$ is the entropy (not to be confused with the enstrophy!). For a polytropic gas we have,

$$S = C_v \log\left(\frac{p}{\rho^\gamma}\right),$$

where γ is the adiabatic index and C_v is the specific heat (at constant volume). (You can get the correct dimensions using the sum of logs and adding a characteristic pressure and density term on the end). For the isentropic case, S is constant throughout the whole fluid, so $p/\rho^\gamma = const$. For our 1D case, we have,

$$\partial_t S + u\partial_x S = 0. \tag{27.3}$$

From above we can write $p = p(\rho, S)$, giving,

$$\partial_t p + u\partial_x p - c_s^2(\partial_t \rho + u\partial_x \rho) = 0, \tag{27.4}$$

where c_s is the speed of sound given by $c_s = \sqrt{(\partial p/\partial \rho)_S}$.

Now we would like to write our three equations in characteristic form, that is, we would like to take these pde's and derive a set of ode's from them. Multiplying out momentum equation (27.1)by some function $l_1(x,t)$ and the continuity equation (27.2) by a similar function $l_2(x,t)$, adding these together and combining them with our pressure equation (27.4) we find,

$$p_t + (u + l_2)p_x + \rho l_2(u_t + uu_x) + \rho l_1 u_x + (l_1 - c_s^2)(\rho_t + u\rho_x) = 0. \tag{27.5}$$

Equation (27.3) is already in characteristic form. We now re-write $D_t S = 0$ as,

$$\frac{dS}{dt} = 0 \text{ on the curve } C_0.$$

The curve C_0 is given by,

$$\frac{dx}{dt} = u(x,t),$$

this corresponds to the case $l_1 = l_2 = 0$ in equation (27.5). The idea now is to look for a set of equations where derivatives would only enter in the following combinations $\partial_t + (stuff)\partial_x$, where each equation has the same $stuff$. Now the last term in equation (27.5) implies $l_1 = c_s^2$ since the $\rho_t + u\rho_x$ part would mean our $stuff = u$ which we want to exclude. For p we therefore find,

$$(\partial_t + (u + l_2))p + \rho l_2(\partial_t + (u + \frac{l_1}{l_2})\partial_x)u = 0,$$

which implies, $l_2 = l_1/l_2 \Rightarrow l_2 = \pm c_s$.

Therefore, our three characteristic equations are,

$$\begin{aligned} \frac{dp}{dt} + \rho c_s \frac{du}{dt} &= 0 \text{ on } C_+ : \frac{dx}{dt} = u + c_s, \\ \frac{dp}{dt} - \rho c_s \frac{du}{dt} &= 0 \text{ on } C_- : \frac{dx}{dt} = u - c_s, \\ \frac{dS}{dt} &= 0 \text{ on } C_0 : \frac{dx}{dt} = u. \end{aligned} \tag{27.6}$$

These are the 1D compress Euler equations written in characteristic form, where C_0,C_- and C_+ are the characteristics (or characteristic curves).

27.2 Isentropic Case

Recall, in this case we have $S = const$ for the whole fluid. So S is the same for all time, hence $p = p(s,\rho) = p(\rho)$. We can therefore write the two remaining equations as conservations laws,

$$\frac{d}{dt}\underbrace{\left(\int \frac{c_s(\rho)}{\rho} d\rho + u\right)}_{R_+} = 0 \text{ on } C_+,$$

$$\frac{d}{dt}\underbrace{\left(\int \frac{c_s(\rho)}{\rho} d\rho - u\right)}_{R_-} = 0 \text{ on } C_-,$$

where we have used the chain rule to re-write $dp(\rho)/dt = (\partial_\rho p)_S d\rho/dt = c_s^2 d\rho/dt$. R_+ and R_- are called Riemann Invariants, R_+ being constant on C_+ and R_- constant on C_-. In a sense we have integrated our equations in the isentropic case. However, we haven't found general solutions of the compressible Euler equations since our characteristics depend on u and c_s which depend on R.

27.3 Polytropic Case

For a polytropic isentropic gas we have $p = const\rho^\gamma$ which implies

$$R_\pm = \frac{2c_s}{\gamma - 1} \pm u.$$

27.4 Small Perturbations Around a Uniform State - 1D Sound

Let us return to the generic case 27.6. In a uniform steady state, $u = 0, \rho = \rho_0, p = p_0$. Consider small perturbations to this steady state,

$$\begin{aligned} \rho &= \rho_0 + \rho_1, \\ p &= p_0 + p_1, \\ u &= u_1, \end{aligned}$$

where $\rho_1 \ll \rho_0$ and $p_1 \ll p_0$. We are therefore in a position to linearise our equations. This gives,

$$\begin{aligned} \frac{dp_1}{dt} + \rho_0 c_{s0} \frac{du_1}{dt} &= 0 \text{ on } C_+ : \frac{dx}{dt} = c_{s0}, \\ \frac{dp_1}{dt} - \rho_0 c_{s0} \frac{du_1}{dt} &= 0 \text{ on } C_- : \frac{dx}{dt} = -c_{s0}, \end{aligned}$$

which implies,

$$\begin{aligned} \frac{d}{dt}(p_1 + \rho_0 c_{s0} u_1) &= 0 \text{ on } C_+ , \\ \frac{d}{dt}(p_1 - \rho_0 c_{s0} u_1) &= 0 \text{ on } C_- , \end{aligned}$$

with,

$$\begin{aligned} x &= c_{s0} t + x_0 \text{ on } C_+ , \\ x &= -c_{s0} t + x_0 \text{ on } C_- . \end{aligned}$$

Integrating we find,

$$\begin{aligned} p_1 + \rho_0 c_{s0} u_1 &= F(x - c_{s0} t) \text{ on } C_+ , \\ p_1 - \rho_0 c_{s0} u_1 &= G(x + c_{s0} t) \text{ on } C_- . \end{aligned}$$

If we add these together and multiply by half we have,

$$p_1 = \frac{1}{2}\left[F(x - c_{s0}t) + G(x + c_{s0}t)\right],$$

we can in a similar manner get a relationship for the velocity. F and G are arbitrary functions which are fixed by the initial conditions of p_1 and ρ_1. We note that the solutions have two parts, one propagates to the left, the other to the right. These are of course sound waves.