# Lecture 28

# 1D Gas Dynamics

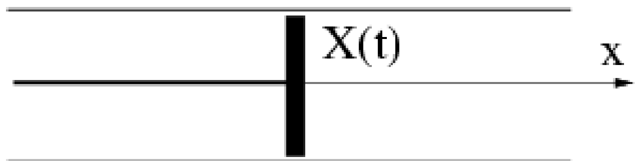


Figure 28.1: Piston

Consider a piston filled with a compressible fluid, figure 28.1. The position of the piston is given by a coordinate $X(t)$ which is a given function. From this we would like to determine the behaviour of the density, pressure and velocity of our compressible fluid. To do this we will draw the characteristics of this problem based on the theory we developed in the previous lecture. Firstly, we assume,

$$X(t) = 0, \text{ for } t = 0,$$
$$X(t) < 0, \text{ for } t > 0.$$

The reason we pull the piston out, instead of pushing it in, will become apparent later. Recall from the previous lecture we have the following equation for entropy,

$$\frac{dS}{dt} = 0 \text{ along characteristics } C_0 : \frac{dx}{dt} = u.$$

One of these trajectories is obviously the piston itself. Fluid particles here will move with the piston. However, far from the piston the fluid hasn't had time to react. Figure 28.2 shows curves of constant entropy. $S$ is conserved along the curves. In fact, the value of $S$ is the same for all the curves because the initial conditions correspond to a steady state.

$$\underbrace{S(p,\rho)}_{t>0} = \underbrace{S(p_0,\rho_0)}_{t=0} = const.$$

Recall, if $S = const$ we have Riemann invariance (Polytropic gas, isentropic fluid) which implies,

$$S = C_v \log\left(\frac{p}{\rho^\gamma}\right).$$

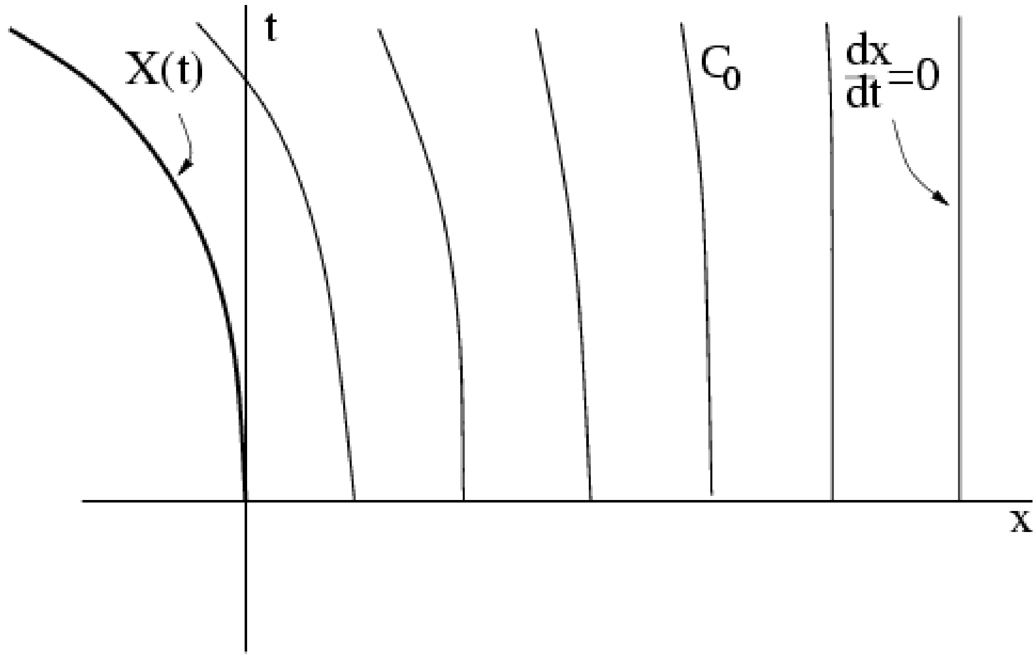


Figure 28.2: Curves of constant Entropy

Therefore,

$$R_{\pm} = \frac{2c_s}{\gamma - 1} \pm u = const \text{ on } C_{\pm} : \frac{dx}{dt} = u \pm c_s.$$

It would be instructive to sketch the curves $C_+$ and $C_-$, figure 28.3. Notice that the slope of $C_-$ is less than $C_0$ while the slope of $C_+$ is greater than it. Firstly we shall consider $R_-$, originating from the $x$-axis. At $t = 0$ we have,

$$R_- = \frac{2c_{s0}}{\gamma - 1} = 0 \text{ with } u = 0,$$

with

$$c_s = c_s(\rho_0, p_0) = c_{s0} = const.$$

The value of $R_-$ is conserved in the whole space between the piston and $\infty$, even for $t > 0$. So we have,

$$\frac{2c_s}{\gamma - 1} - u = \frac{2c_{s0}}{\gamma - 1} = const. \tag{28.1}$$

This example leads is a useful definition,

**Definition**: Continuous 1D motions with $S = const$ and one Riemann invariance constant are called *simple waves.*

Now for $R_+$. We see for $C_+$ there are two regions, some of the characteristics start on the $x$-axis while others start on the edge of the piston. These two regions are divided by the $C_{+0}$.

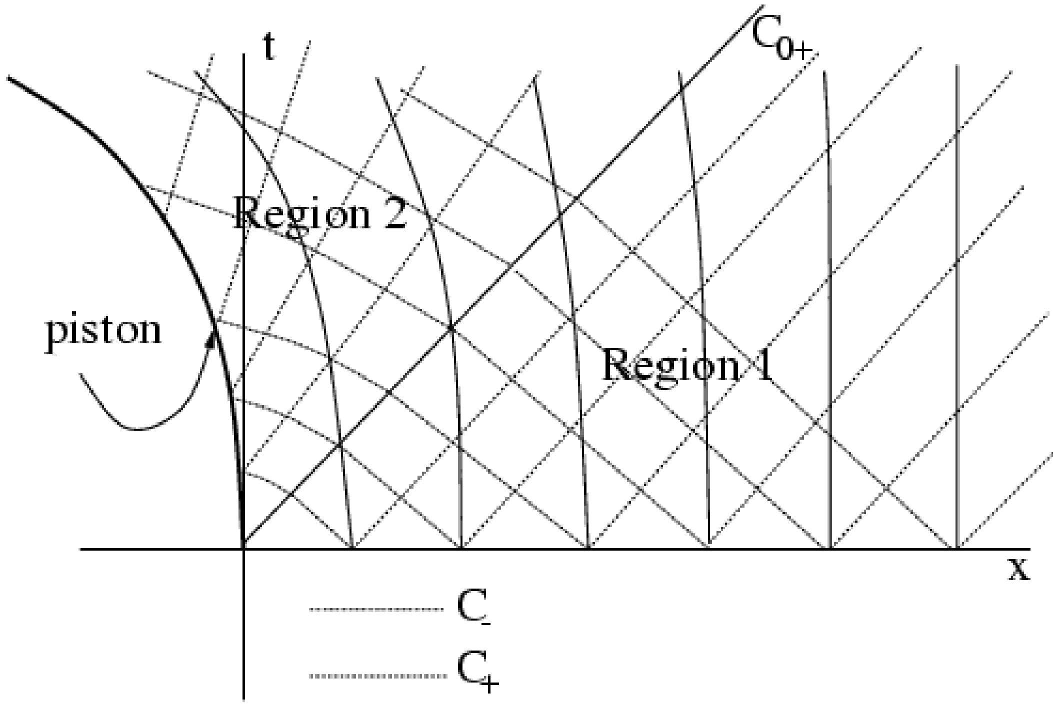


Figure 28.3: $C_{+}andC_{-}curves$

**Region 1** (on $x$-axis): we have,

$$R_{+} = \frac{2c_s}{\gamma - 1} + u = \frac{2c_{s0}}{\gamma - 1}. \tag{28.2}$$

Taking equation (28.1) and subtracting equation (28.2) we find that $u = 0$ in region 1. This means that in this region we have undisturbed motion for all $t$. Therefore, all the characteristics are straight lines

**Region 2** (at piston): Here we have,

$$R_{+} = \frac{2c_s}{\gamma - 1} + u = const \text{ on } C_{+} : \frac{dx}{dt} + c_s.$$

Combining this with equation (28.1) (which, recall, is valid everywhere) we find,

$$2u = -\frac{2c_{s0}}{\gamma - 1} + const \text{ on } C_{+} : \frac{dx}{dt} = u + c_s.$$

This tells us that the velocity remains constant on these characteristics. Adding equation (28.1) to this, we also find that $c_s$ is constant.

$$\frac{4c_s}{\gamma - 1} = \frac{2c_{s0}}{\gamma - 1} + const.$$

So again we find these characteristics are straight lines,

$$C_{+} : \frac{dx}{dt} = u + c_s = c_{s0} + \frac{\gamma + 1}{2} u.$$

We can integrate this equation to obtain,

$$x = X(\tau) + \left[c_{s0} + \frac{\gamma+1}{2}\dot{X}(\tau)\right](t-\tau), \tag{28.3}$$

where $\tau$ parameterises the curves $C_+$. When $t = \tau$, $x = X(\tau)$, therefore $u = \dot{X}(\tau)$ which implies,

$$c_s = c_{s0} + \frac{\gamma+1}{2}\dot{X}(\tau),$$

where $\tau$ is given implicitly by equation (28.3).

So why did we decide to pull the piston out as instead of pushing it in? Pushing the piston would always produce a shock. Similarly if we decelerate the piston, while pulling it out, we would create a caustic or an intersection of characteristics.