Lecture 2

Ideal Fluids

2.1 What is an Ideal Fluid?

An ideal fluid:

- *Incompressible*, $\rho = const$
- *Inviscid* - i.e no dissipation. The momentum is only transfered via pressure and not due to internal friction (viscosity).

2.2 The Ideal Fluid Equations

2.2.1 Derivation - Part A

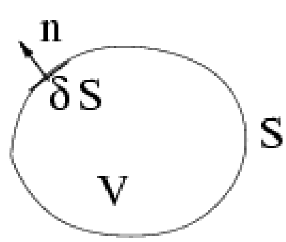

Figure 2.1: A small volume V bounded by a *fixed* surface S

Consider a small volume V bounded by a *fixed* surface S, figure 2.1. If $\mathbf{n}$ is the unit vector perpendicular to a surface element δS, then the volume of fluid leaving δS in unit time is

$$u_n \delta S, \tag{2.1}$$

where $u_n = \mathbf{u} \cdot \mathbf{n}$ is the component of velocity $\mathbf{u}$ normal to the surface element δS. Therefore, the net volume of fluid leaving through the whole surface S is, using Gauss's theorem

$$\int_S \mathbf{u} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{u} \, dV = 0. \tag{2.2}$$

We have used the fact that our fluid is incompressible, i.e. that the last integral above must be zero for any volume V. This means the integrand itself must be zero. Let us assume that $\mathbf{u}$ varies smoothly in space and let us also choose our volume V to be very small such that $\nabla \cdot \mathbf{u}$ is almost constant in this volume. Then,

$$\int_V \nabla \cdot \mathbf{u} dV \simeq V \nabla \cdot \mathbf{u} = 0.$$

The resulting equation is the *Incompressibility Condition* (otherwise known as the *Continuity Equation*).

$$\nabla \cdot \mathbf{u} = 0. \tag{2.3}$$

Remark: We have not used the fact that the fluid is inviscid, so the incompressibility condition is the same for viscous fluids.

2.2.2 Derivation - Part B

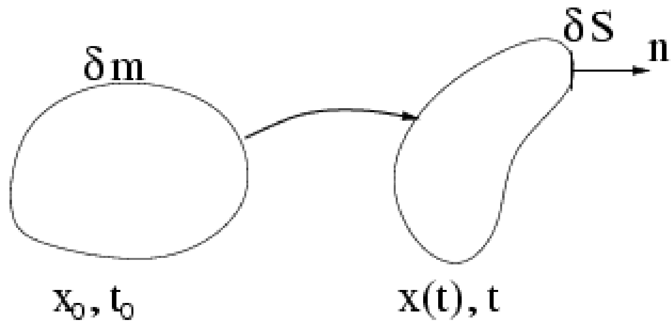

Figure 2.2: Fluid Blob

Consider a small fluid blob, figure 2.2. According to Newton's Second Law its acceleration $D_t \mathbf{u}$ multiplied by its mass δm is equal to the net force acting on it. Therefore,

$$m D_t \mathbf{u} = net\, force, \tag{2.4}$$

where $m = \int_V \rho dV \simeq \rho V$ for smooth ρ and small V. Now, $D_t \mathbf{u}$ is the rate of change of velocity of the moving blob, rather than the time derivative at a fixed point $\mathbf{x}$.

$$\begin{aligned} D_t \mathbf{u} &= \frac{d}{dt} \mathbf{u}(\mathbf{x}(t), t) \qquad (2.5) \\ &= \partial_t \mathbf{u} + \frac{dx}{dt} \partial_x u + \frac{dy}{dt} \partial_y v + \frac{dz}{dt} \partial_z w \\ &= \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}. \end{aligned}$$

The operator $\boxed{D_t = \partial_t + (\mathbf{u} \cdot \nabla)}$ is called the *Material or Convective Derivative.*

The *net force* = *pressure force* + *gravity.* So we need expressions for the pressure force and gravity. The pressure force is

$$pressure force = -\int_S p \mathbf{n} dS = -\int_V \nabla p dV, \tag{2.6}$$

which is approximately $-V\nabla p$ because the volume V is small and ∇p is almost the same everywhere in this volume. Gravity has the following formalisation,

$$gravity = m\mathbf{g} = \rho V\mathbf{g}. \tag{2.7}$$

Therefore, our equation becomes,

$$\begin{aligned} \rho V D_t\mathbf{u} &= -V\nabla p + \rho V\mathbf{g} \\ \implies D_t\mathbf{u} \equiv \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\frac{\nabla p}{\rho} + \mathbf{g}. \end{aligned} \tag{2.8}$$

This is the *Momentum Equation.* The Momentum Equation and the Incompressibility condition make up the *Euler Equations*[1].

[1]However, the Momentum Equation is often called the Euler Equation on its own.