Lecture 3

Bernoulli's Theorem and Vorticity

Let us continue considering an ideal fluid.

3.1 Other Forms of the Euler Equations

Letting $\lambda = gz$ we can include the gravitational effects with our pressure term.

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla(\frac{p}{\rho} + \lambda).$$

However, we can also use the vector identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2}\nabla \mathbf{u}^2,$$

to rewrite this in the following form,

$$\partial_t \mathbf{u} + (\nabla \times \mathbf{u}) \times \mathbf{u} = -\nabla(\frac{p}{\rho} + \frac{\mathbf{u}^2}{2} + \lambda).$$

We now define an important new quantity, the curl of the velocity field.

Definition: $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is called *vorticity.*

Therefore, our Euler equation above can be re-written as

$$\partial_t \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} = -\nabla(\frac{p}{\rho} + \frac{\mathbf{u}^2}{2} + \lambda). \tag{3.1}$$

3.2 Steady Flow and Bernoulli's Theorem

Consider the above equation (3.1) for the case of a steady flow, that is, all our partial time derivatives are zero. We can re-write our equation as

$$\boldsymbol{\omega} \times \mathbf{u} = -\nabla H, \tag{3.2}$$

where

$$H = \frac{p}{\rho} + \frac{\mathbf{u}^2}{2} + \lambda.$$

If we now multiply equation (3.2) by $\mathbf{u}\cdot$ we discover that

$$(\mathbf{u} \cdot \nabla)H = 0.$$

In fact, if we recall our definition of the material derivative and remembering that we are talking about a steady flow, we can write the above as

$$D_t H = 0. \tag{3.3}$$

Therefore, the quantity H is conserved along the fluid's streamlines. This is known as *Bernoulli's Theorem.*

Bernoulli's Theorem: In a stationary ideal fluid, H is constant along a fluid's streamlines.

Remark: H in general will have different values for different streamlines.

3.3 A Steady Irrotational Flow

Recall an *irrotational* flow is one with zero vorticity, that is

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0.$$

Therefore, in this case, equation (3.1) becomes

$$\nabla H = 0.$$

This means, that for an irrotational steady flow, the constant H has the same value for all streamlines. The reverse statement is also true - if H is the same for all streamlines then the vorticity $\boldsymbol{\omega} = 0$. But, H is given by its value far upstream. Therefore, the vorticity in a steady flow, if zero upstream, will remain zero everywhere. This is always true, unless the streamlines pass by a solid boundary (where ν is always important) and then return to the bulk of the fluid. This phenomenon, called *flow separation*, will be studied later in this course.

3.4 Examples

3.4.1 Contracting Flow in a Pipe

Consider figure 3.1. We can apply Bernoulli's theorem to this contracting flow as follows. Firstly, we notice that the vertical co-ordinates at the beginning and end of the contraction are approximately equal. Therefore, we can neglect the gravitational term from Bernoulli's Theorem. Secondly, we can make the observation that the velocity of

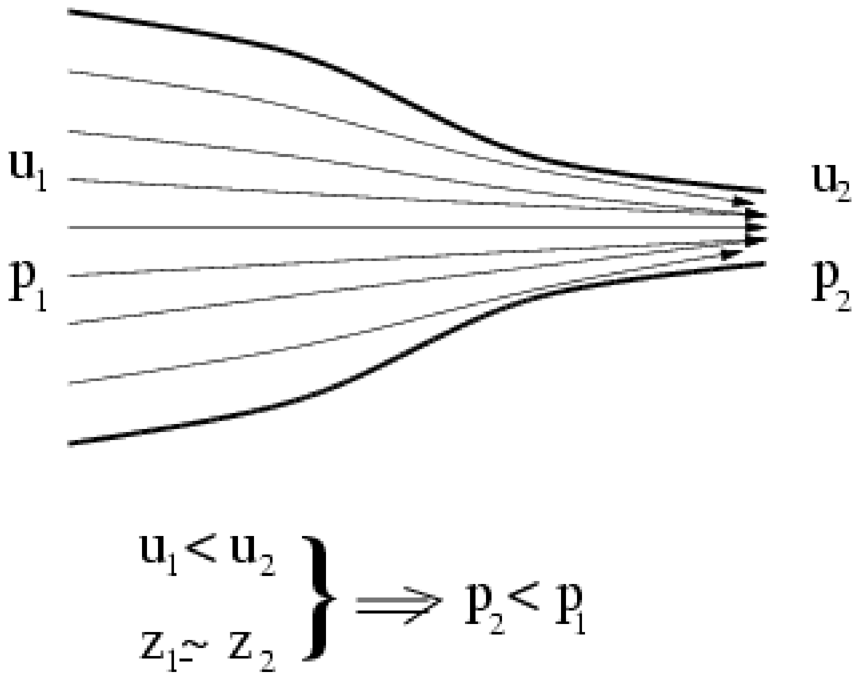

Figure 3.1: A Contracting Flow in a Pipe

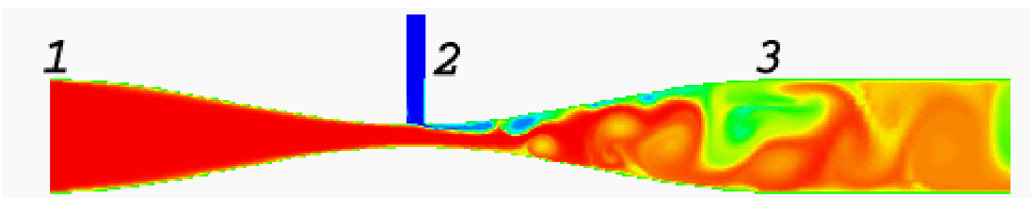

Figure 3.2: A Venturi Pipe

the fluid is greater at the narrow end of the pipe. (Just think about squeezing the end of a garden hose!). Therefore, since H is a constant, Bernoulli tell us that the pressure is lower in the contracted part of the pipe.

In fact, this simple analysis can be extended to gain an understanding of a Venturi Pipe. A Venturi Pipe, figure 3.2, is a small device often found in yachts. It is used to suck unwanted water out of the bottom of a boat. There is an animation of this flow in the online supplement to this lecture on *MathStuff*. Figure 3.2 shows a still from this animation. One can generalise our contracting pipe example to that of the Venturi pipe, and therefore deduce that the pressure is lower in region 2 than in the other two regions. This low pressure sucks the water out of the bottom of the boat. We should note that this solution, (i.e. an extension of the contracting pipe example and assuming an ideal fluid) is absolutely symmetric around region 2. However, in reality the flow is very assymetric because of flow separation, as you can see in our figure. Flow separation occurs when viscosity becomes important in the expansion section. That is, the fluid is no longer ideal in this region.

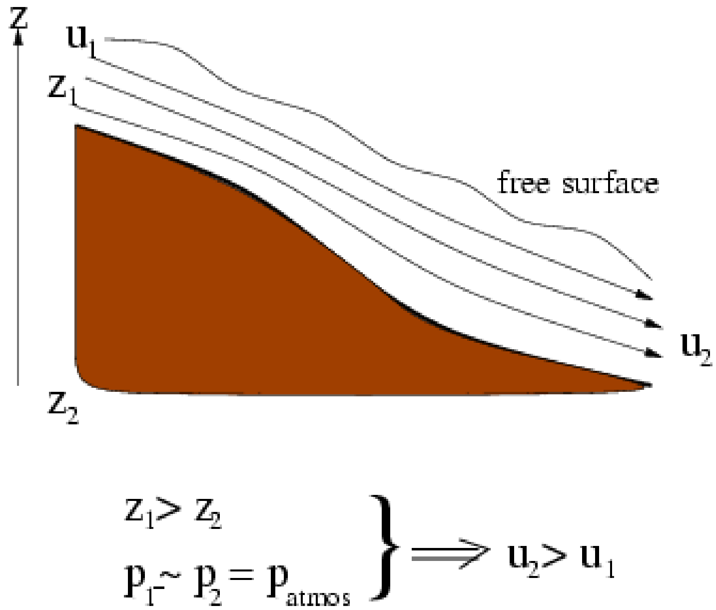

Figure 3.3: Steady Flow Down a Slope

3.4.2 Steady Flow Down a Slope

Consider a steady flow down a slope, figure 3.3. In this case we realise that the pressure at the top and bottom of the slope are approximately equal (i.e. atmospheric pressure!). Since the slope is obviously higher at the top of the slope, we deduce from Bernoulli's theorem that the fluid velocity is greater at the bottom.

3.4.3 Lift on an Aerofoil

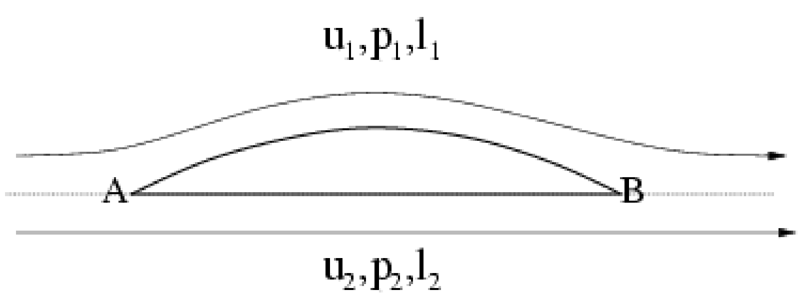

Figure 3.4: Lift on an Aerofoil

Consider an aerofoil, figure 3.4. The top of the aerofoil is longer than the bottom, $l_1 > l_2$. Hence, we could say that fluid elements, travelling from points A to B, going over the top of the aerofoil have further to travel than those going under it. Therefore, for all the fluid particles leaving A to arrive simultaneously at B, those travelling over the aerofoil must be moving faster. That is $u_1 > u_2$, and so using Bernoulli we deduce that $p_1 < p_2$. This pressure difference is what produced the lift, (the net force pointing upwards). This arguement seems to work, but is actually flawed! So what is wrong with it?

3.5 For a Time Dependent Irrotational Flow

For an irrotational flow (i.e one with zero vorticity) the velocity field can be written in terms of a potential ϕ,

$$\mathbf{u} = \nabla\phi, \tag{3.4}$$

where $\phi = \phi(\mathbf{x}, t)$. Substituting this into the Euler equation, of form (3.1), and removing ∇ on both sides, via integration, we find

$$\partial_t\phi + \frac{p}{\rho} + \frac{1}{2}(\nabla\phi)^2 + \lambda = f(t).$$

One can assume $f(t) = const$ without loss of generality. This is because ϕ is defined only up to an arbitrary function of time t, which can always be chosen so that $f(t) = const$. Therefore,

$$\partial_t\phi + \frac{p}{\rho} + \frac{1}{2}(\nabla\phi)^2 + \lambda = const. \tag{3.5}$$

This is known as *Bernoulli's Theorem* for a *Time Dependent Irrotational Flow.*

3.6 Solving Bernoulli's Theorem

Although equation (3.5) is a partial differential equation which involves ϕ, the function ϕ **is not** usually found by solving it. Instead, ϕ is found from the incompressibility condition (2.3),

$$\nabla \cdot \mathbf{u} = \nabla^2\phi = 0.$$

This should be familiar, it is *Laplace's Equation.* Notice that this equation does not involve ∂_t, therefore any time dependence of ϕ can only arise from the time dependence of the BC's. With time independent BC's we return to the previous example of a stationary irrotational flow. However, equation (3.5) is not useless, and in fact is commonly used to find the pressure p when ϕ is known.