Lecture 4

Conservation Laws for Ideal Fluids

4.1 Energy Conservation

Applying $\int \mathbf{u}\cdot$ to the Euler equation (3.1) we find the following

$$\frac{1}{2}\partial_t \int \mathbf{u}^2 d\mathbf{x} = -\int \mathbf{u}\cdot\nabla H d\mathbf{x}.$$

We have eliminated the second term on the LHS of the Euler equation (the advection term) because it is now zero. We have also pulled the partial time derivative outside the integral since it is independent of the spatial integration. However, since we are dealing with an incompressible fluid we can simplify the above equation using the incompressible condition and one of our vector identities.

$$\frac{1}{2}\partial_t \int \mathbf{u}^2 d\mathbf{x} = -\int \nabla\cdot(\mathbf{u}H) d\mathbf{x} = 0.$$

Now, multiplying by our density ρ, which is a constant, we finally arrive at,

$$\partial_t E = 0, \tag{4.1}$$

where,

$$E = \frac{1}{2}\rho \int \mathbf{u}^2 d\mathbf{x}. \tag{4.2}$$

E is called the total kinetic energy of the fluid. Therefore, we conclude that, in an ideal fluid, the total kinetic energy is conserved.

Remark: By analogy with discrete mechanics (fluid mechanics deals with continuous media) one would expect conservation of the total energy, i.e. the kinetic energy and the potential energy. Why then do we conclude that the total kinetic energy is conserved by itself? The answer is that in an ideal fluid the total potential energy cannot change!

$$\int_V g\rho z d\mathbf{x} = g\rho \int_V z d\mathbf{x} = constant.$$

4.2 Vorticity

4.2.1 The Vorticity Equation

Taking the curl of both sides of the Euler equation (3.1) we obtain,

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = 0. \tag{4.3}$$

This is known as the *Vorticity Equation.* However, using some vector calculus we can reformulate it,

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\omega \cdot \nabla)\mathbf{u},$$

or in terms of the material derivative,

$$D_t \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}. \tag{4.4}$$

The RHS of this equation is called the *Vortex Stretching* term; it plays an important role in many practical examples.

Note: The vorticity formulation of a fluid does not involve p.

4.2.2 The 2D Limit

Consider the 2D limit of the vorticity equation (4.4). That is,

$$\begin{aligned}\mathbf{u} &= (u, v, 0) = \mathbf{u}(x, y, t),\\ \omega &= (0, 0, \Omega),\\ \Omega &= \Omega(x, y, t).\end{aligned}$$

Then, in this case the stretching term is zero.

$$(\omega \cdot \nabla)\mathbf{u} = 0.$$

Now, the z-component of equation (4.4) has become

$$\boxed{D_t \Omega = 0}.$$

Therefore, we conclude that the vorticity within any fluid element is conserved. We say that "*Vorticity is a Lagrangian Invariant in 2D*".

Note: Vorticity is not conserved in 3D because the stretching term is generally non-zero.

It is easy to see that any function f of vorticity is a Lagrangian invariant, i.e.

$$D_t f(\Omega) \equiv (\partial_t + (\mathbf{u} \cdot \nabla)) f(\Omega) = 0. \tag{4.5}$$

(Just use the chain rule to check this). Now, because $\nabla \cdot \mathbf{u} = 0$, one can also re-write (4.5) as,

$$\partial_t f + \nabla \cdot (\mathbf{u} f) = 0.$$

Integrating this equation over the 2D plane we find,

$$\partial_t \int f(\Omega) dxdy = -\int \nabla \cdot (\mathbf{u} f) dxdy = -\oint_C \mathbf{u} \cdot \mathbf{n} f dxdy,$$

where in the last integral C is the boundary (or ∞ for infinite plane). The normal component of $\mathbf{u}$ should be zero on C, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$. Therefore, we find that

$$\partial_t \int f(\Omega) dxdy = 0.$$

This means that $\boxed{\int f(\Omega) dxdy}$ is a *Motion Integral* (i.e. it is conserved) for any f.

4.2.3 Special Cases

$$I_n = \int \Omega^n dxdy.$$

Integrals I_n are called the *Enstrophy Series of Invariants*. In particular, I_2 is known as the *Enstrophy*.

$$I_2 = \int \Omega^2 dxdy.$$

Later, we will consider 2D vortex dynamics in detail. In this lecture we are only concerned with conservation laws. In the 3D case the vortex stretching term is non-zero. This prevents us from deriving a similar vorticity conservation equation, as we found for our 2D system. However, there are still conservation laws in 3D dynamics. In the next couple of lectures we will consider two such theorems, which were derived by Kelvin and Helmholtz.