Lecture 5

Kelvin's Circulation Theorem

5.1 Kelvin's Circulation Theorem

5.1.1 Theorem

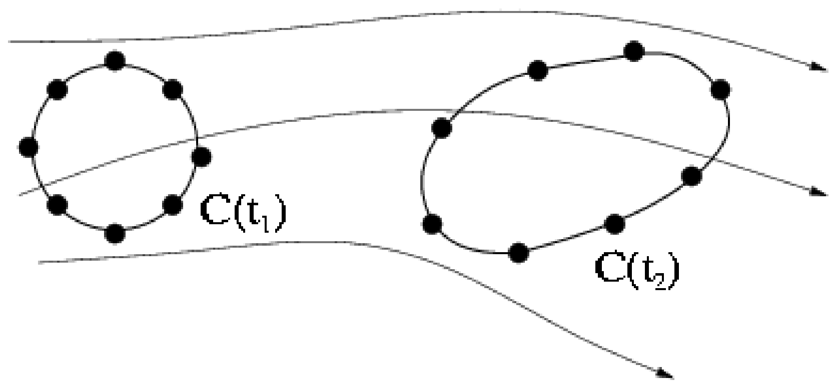

Figure 5.1: Kelvin's Circulation Theorem

Let an inviscid incompressible fluid of constant density be in motion in the presence of a conservative body force $\mathbf{g} = -\nabla\lambda$ per unit mass. Let $C(t)$ denote a closed circuit that consists of the same fluid particles as time proceeds, figure 5.1. Then the circulation

$$\Gamma = \int_{C(t)} \mathbf{u} \cdot d\mathbf{x},$$

round $C(t)$ is independent of time.

5.1.2 Proof

This is really rather short, if we appeal to the following lemma:

$$\frac{d}{dt}\int_{C(t)} \mathbf{u} \cdot d\mathbf{x} = \int_{C(t)} D_t\mathbf{u} \cdot d\mathbf{x}.$$

You will be asked to prove this on one of the remaining example sheets. Then, using Euler's equation, we obtain,

$$\frac{d\Gamma}{dt} = -\int_{C(t)} \nabla\left(\frac{p}{\rho} + \lambda\right) \cdot d\mathbf{x} = -\left[\frac{p}{\rho} + \lambda\right]_C,$$

where the notation $[\ ]_C$ denotes the change after one loop of C. However, the change in the quantity $p/\rho + \lambda$ is zero since p, ρ and λ are all single valued functions of position. Hence, we've proved our theorem.

5.1.3 Further Observations about this Theorem

Observation 1 - C denotes a "dyed" circuit, composed of the same fluid particles as time proceeds; the result is not true in general if C is a closed curve fixed in space.

Observation 2 - The conditions of incompressibility and constant density are not essential: Kelvin established his result subject to weaker conditions.

Observation 3 - The theorem does not require the fluid region to be simply connected, i.e. it does not require the dyed circuit C to be spannable by a surface S lying wholly in the fluid. (You should remember this property when we consider vortex shedding off an aerofoil later in this lecture).

Observation 4 - The inviscid equations of motion enter the proof only in helping to evaluate a line integral round C, so if viscous forces happened to be important elsewhere in the flow, i.e. off the curve C, this would not affect the conclusion that Γ remains constant round C.

5.2 Trailing Vortices and Lift on an Aerofoil

The shedding of a starting vortex is essential to the generation of lift on an aerofoil. Kelvin's theorem can be used to understand why this is so. Consider figure 5.2. The situation is that, at time t after the aerofoil started moving, the viscous forces and vorticity will be confined to:

(i) a thin boundary layer on the aerofoil,

(ii) a thin wake,

(iii) the rolled-up "core" of the starting vortex.

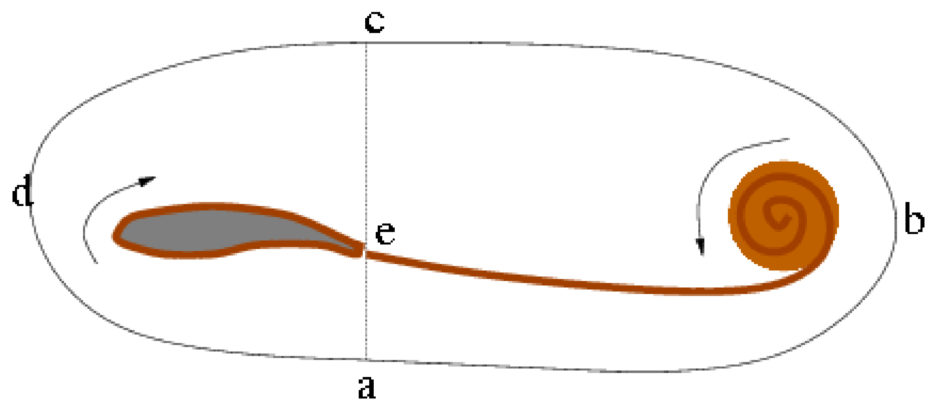

Figure 5.2: Vortex Shedding by an Aerofoil

Consider now a dyed circuit *abcda* which is large enough to have been clear of all these regions since the start of the motion. As the original state was one of rest, the circulation round that circuit was originally zero. Thus, by Kelvin's theorem, the circulation round that circuit will still be zero for time t. (c.f observation 4 above). Therefore, if we sketch in a line *aec* - an instantaneous line in space at time t such that the curve *aecda* encloses the aerofoil but not the wake or the starting vortex - then the circulation round *aecda* must be equal and opposite to that around *abcea*. You might now see where this argument is going. What we now observe is that, as the aerofoil starts to move, positive vorticity is shed in the form of a starting vortex. By Stokes theorem we have,

$$\int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = \int_C \mathbf{u} \cdot d\mathbf{x},$$

which gives a positive circulation round *abcea*. Thus, from our proceeding argument, this implies that there must be negative circulation round *aecda*. This has been observed experimentally. The vortex shedding continues until the circulation round the aerofoil is sufficient to make the main irrotational flow smooth at the trailing edge. At this stage no further vorticity is shed into the wake from the boundary layers on the upper and lower surfaces of the aerofoil. Thereafter the aerofoil retains its final "Kutta-Joukowski" value of the circulation.

5.3 The Persistence of Irrotational Flow

5.3.1 Cauchy-Lagrange Theorem

Let an inviscid incompressible fluid of constant density move in the presence of a conservative body force. Then if a portion of the fluid is initially in irrotational motion, that portion will always be in irrotational motion.

5.3.2 Proof

We will prove this theorem by contradiction. We suppose that the vorticity were *not* identically zero throughout that portion of fluid at a later time. By virtue of Stokes theorem we have,

$$\int_C \mathbf{u} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS,$$

and it would be then possible to select some small closed dyed circuit around which the circulation would be non-zero. However, this would violate Kelvin's theorem, because the circulation round such a circuit must initially have been zero, on account of Stokes theorem and the fact that $\nabla \times \mathbf{u}$ was initially zero. Our initial assumption must therefore be false. This completes our proof by contradiction.

5.3.3 Further Observations

2D Case - For 2D flows the result is obvious from the vorticity equation.

$$D_t \omega = 0.$$

If ω is zero for a portion of the fluid at time $t = 0$, then ω will remain zero for each fluid element constituting that portion for all time t.

3D Case - For 3D flows this result is not obvious. In this case we have the vorticity equation,

$$D_t \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}.$$

It is here that our theorem comes into its own.

Irrotational flows are important, then, even in 3D. The velocity field can then be written as,

$$\mathbf{u} = \nabla \phi, \tag{5.1}$$

and ϕ will be a single-valued function of position when the flow region is simply connected. As the fluid is incompressible, $\nabla \cdot \mathbf{u} = 0$, so ϕ satisfies Laplace's Equation. The general theory of irrotational flow is a classical and important part of fluid dynamics. However, in the following lectures we will consider fluid motions in which the vorticity is *not* zero. In such a case, there is no such thing as a velocity potential ϕ and $\mathbf{u}$ cannot be written as in equation (5.1).