Lecture 6

Helmholtz Vortex Theorem

6.1 Vortex Lines and Vortex Tubes

A Vortex Line - is a curve, at any particular time t, which has the same direction as the vorticity vector,

$$\boldsymbol{\omega} = \nabla \times \mathbf{u},$$

at each point. A more mathematical description is that a vortex line $x = x(s)$, $y = y(s)$, $z = z(s)$, is obtained by solving,

$$\frac{dx/ds}{\omega_x} = \frac{dy/ds}{\omega_y} = \frac{dz/ds}{\omega_z},$$

at a particular time t.

A Vortex Tube - The vortex lines which pass through some simple closed curve in

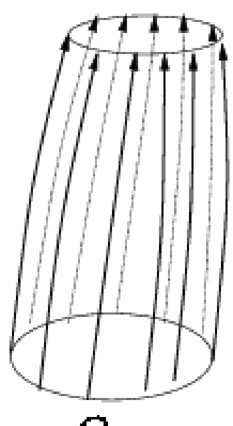

Figure 6.1: Vortex Tube

space are said to form the boundary of a *vortex tube*, figure 6.1.

6.2 Helmholtz Theorem

Suppose now that we have an inviscid, incompressible fluid of constant density moving in the presence of a conservative body force (so that Kelvin's Circulation Theorem holds). Then,

(i) *The fluid elements that lie on a vortex line at some instant continue to lie on a vortex line, i.e. vortex lines "move with the fluid".*

(ii) *An immediate consequence of this is that vortex tubes move with the fluid in a like manner. So the quantity,*

$$\Gamma = \int_S \boldsymbol{\omega} \cdot \mathbf{n}\, dS,$$

is the same for all cross-sections S of a vortex tube. Furthermore, Γ is independent of time.

The quantity Γ is therefore a conserved property of the tube as a whole, called the *tube strength.*

6.3 Proof

6.3.1 Part (i)

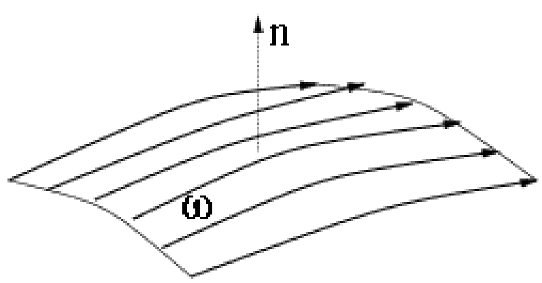

Figure 6.2: Vortex Surface

We first define a *vortex surface* as a surface such that $\boldsymbol{\omega}$ is tangent to the surface at every point, figure 6.2. The proof proceeds by viewing the vortex line, in its initial configuration, as the intersection of two vortex surfaces. Mark the particles which occupy one of the vortex surfaces , at $t = 0$, with dye. Consider a closed circuit C made up of a particular set of dyed particles and spanned by a portion S_* of the vortex surface. At time $t = 0$ the circulation round C is zero, since by Stokes theorem,

$$\int_C \mathbf{u} \cdot d\mathbf{x} = \int_{S_*} \boldsymbol{\omega} \cdot \mathbf{n} dS,$$

and $\boldsymbol{\omega} \cdot \mathbf{n}$ is zero on S_*. Now, as time proceeds the dyed sheet of fluid will deform, but the circulation round C will remain zero, by Kelvin's Circulation Theorem. This being so for all circuits such as C it follows, by use of Stokes theorem again, that $\boldsymbol{\omega} \cdot \mathbf{n}$ will remain zero at all points of the dyed sheet of fluid. That sheet therefore remains a vortex surface as time proceeds. The proof is completed by noting that the intersection of two such dyed sheets therefore remains the intersection of the two vortex surfaces, i.e. it remains a vortex line.

6.3.2 Part (ii)

The statement that Γ is independent of the cross-section S has nothing to do with the equations of motion but is simply a consequence of the fact that the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is divergence-free. The statement that Γ is independent of time follows on considering a circuit, such as C_1 in figure 6.1, composed of fluid particles which lie on the wall of the vortex tube and encircle it. By Stokes theorem, Γ is the circulation round C_1, and by Kelvin's Circulation Theorem this remains constant as time proceeds.

6.4 Tornadoes and Cups of Tea

Now you might think that tornadoes and cups of tea have very little in common! However, they are both great examples of Helmholtz theorem at work. In particular these are both examples of vortex tube stretching and shortening.

The General Case - Consider a *thin* vortex tube in which $\boldsymbol{\omega}$ is virtually constant across any particular cross-section. In this case Γ is essentially just the product of $\omega\delta S$, where δS is the normal cross-section of the tube. But, δS is also the normal cross-section of the fluid continually occupying the tube, and since the fluid must conserve its volume, δS will vary inversely with the length l of a small section of the tube. Thus the vorticity ω varies in proportion to l; stretching of the vortex tube by the fluid motion intensifies the local vorticity, while shortening (widening) of the tube will have the opposite effect.

Tornadoes - In a tornado the strong thermal up-draughts into the thunderclouds overhead produce intense stretching of vortex tubes, and hence the potentially devastating rotary motions observed. The funnel cloud we observe is a direct marker of the vortex tube. This cloud is due to the extreme low pressures which are found in the core of the vortex, which in turn is the location of the vorticity. As the the thundercloud moves on the funnel cloud tips over and gets stretched, intensifying the vorticity inside, figure 6.3.

The "spin down" of a cup of tea - is an example of the opposite case where we observe the shortening of a vortex tube. The main body of the fluid is essentially inviscid and in rapid rotation, the centrifugal force being (almost) balanced b a radially inward pressure gradient. This pressure gradient also imposes itself throughout the thin viscous boundary layer on the bottom of the cup, where it is stronger then required, for the fluid in the boundary layer rotates much less rapidly. That fluid therefore spirals inward (this can be observed by noting that tea leaves on the bottom of the cup congregate in the middle)

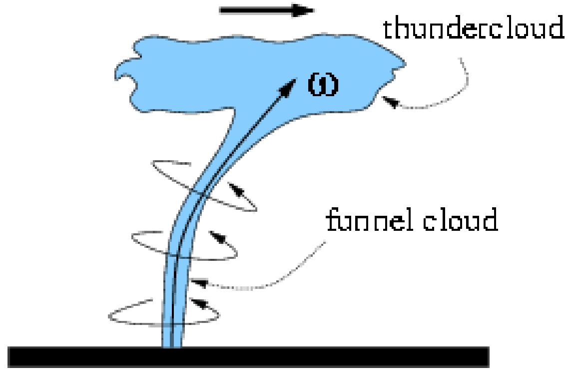

Figure 6.3: A Tornado

and eventually turns up and out of the boundary layer, as in figure 6.4. In this way

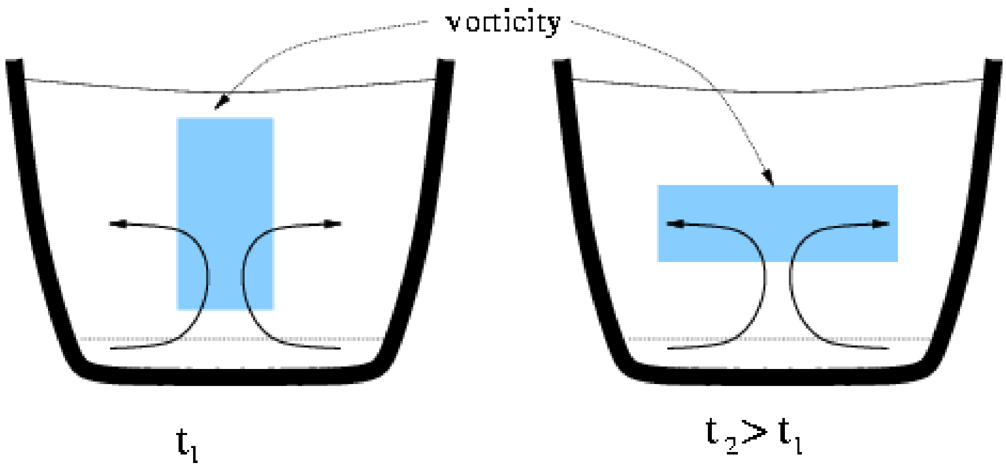

Figure 6.4: The spin down of a cup of tea

vortex tubes in the main body of the fluid become shorter and expand in cross-section, so that the vorticity decreases with time. It is by this subtle mixture of inviscid and viscous dynamics that the apparently fast spin down of a stirred cup of tea is achieved.