Lecture 7

2D Ideal Fluids

7.1 The Streamfunction

Consider the 2D vorticity equation,

$$(\partial_t + (\mathbf{u} \cdot \nabla))\Omega = 0, \tag{7.1}$$

where,

$$\begin{aligned}
\Omega &\equiv \boldsymbol{\omega}_z = \partial_x v - \partial_y u, \\
\boldsymbol{\omega} &= \nabla \times \mathbf{u}, \\
\mathbf{u} &= (u, v, 0).
\end{aligned}$$

One can introduce a *Streamfunction* ψ such that

$$\begin{aligned}
u &= \partial_y \psi, \\
v &= -\partial_x \psi.
\end{aligned}$$

You should check that $\nabla \cdot \mathbf{u} = 0$ is satisfied automatically in this case. We can also write the vorticity in terms of the streamfunction via,

$$\Omega = -\nabla^2 \psi.$$

7.2 Simple Solutions to the 2D Vorticity Equation

7.2.1 A Shear Flow

Consider a shear flow, figure 7.1. In general, our shear flow has a solution of the form,

$$\mathbf{u} = (f(y), 0), \tag{7.2}$$

where f is some function of y. Indeed, $\Omega = -f'(y)$ is a function of y only. So,

$$\begin{aligned}
&\partial_t \Omega = 0, \\
&(\mathbf{u} \cdot \nabla)\Omega = u\partial_x \Omega + v\partial_y \Omega = 0, \\
\Rightarrow &D_t \Omega = 0.
\end{aligned}$$

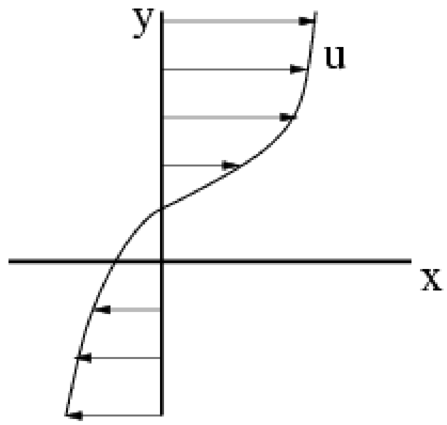

Figure 7.1: Shear Flow

Note 1: The flow is not stable for some $f(y)$. Small initial deviations from the solution may experience an unbounded growth.

Note 2: Viscosity removes arbitrariness in $f(y)$.

We will return to the discussion of these properties later, when we consider stability and viscous flows.

7.2.2 A Circular Vortex

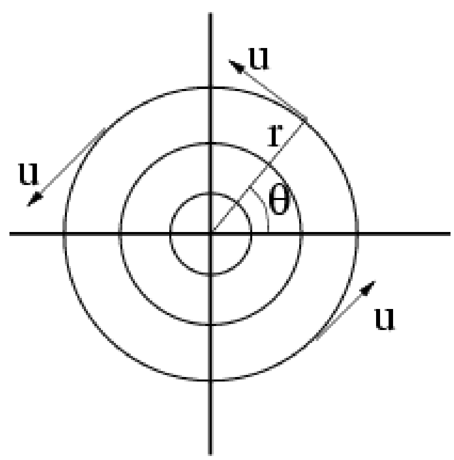

Figure 7.2: Circular Vortex

Consider a circular vortex, figure 7.2. For a such a vortex we have,

$$\Omega = f(r),$$
$$u_r = 0,$$
$$u_\theta = g(r).$$

One can see that,

$$D_t\Omega = \partial_t\Omega + u_r\partial_r\Omega + \frac{u_\theta}{r}\partial_\theta\Omega$$

That is, $\Omega = f(r)$ is a solution to the fluid's equations for any $f(r)$. To find $g(r)$ we use,

$$\begin{aligned}
&\oint_{C(r)} \mathbf{u}\cdot d\mathbf{x} = \int_{S(r)} \boldsymbol{\omega}\cdot\mathbf{n}dS = \int_{s(r)} \Omega dS \\
\Rightarrow &u_\theta \cdot 2\pi r = 2\pi\int_0^r f(r')r'dr' \\
\Rightarrow &u_\theta = g(r) = \frac{1}{r}\int_0^r f(r')r'dr'.
\end{aligned}$$

Special Case - A Rankine Vortex

A Rankine vortex has a velocity profile, figure 7.3, like

$$u_\theta = \begin{cases} Cr, & r < a, \\ \frac{Ca^2}{r}, & r \geq a. \end{cases}$$

This corresponds to a vorticity distribution of

$$\Omega = \begin{cases} 2C & r < a, \\ 0 & r \geq a, \end{cases}$$

see figure 7.4.

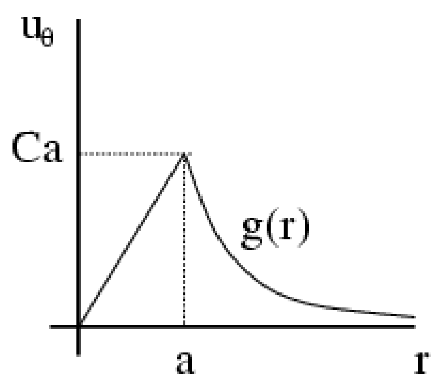

Figure 7.3: Rankine Vortex Angular - Velocity Profile

Special Case - A Point Vortex

A point vortex is another special case, figure 7.5. In this case we find,

$$\begin{aligned}
\Omega &= \Gamma\delta(x)\delta(y), \\
u_\theta &= \frac{\Gamma}{2\pi r}.
\end{aligned}$$

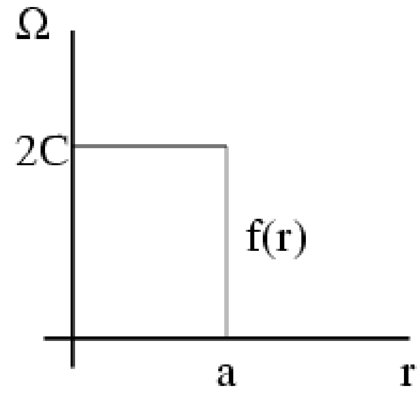

Figure 7.4: Rankine Vortex - Vorticity Profile

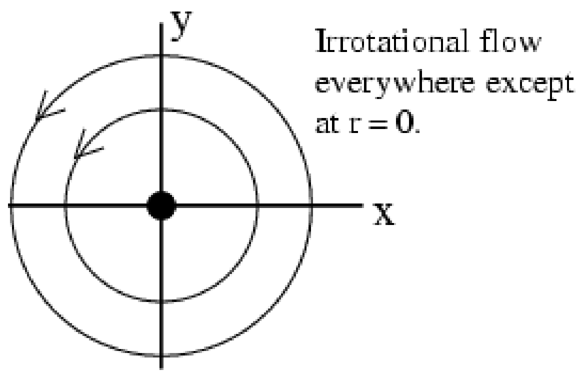

Figure 7.5: Point Vortex

7.2.3 A Vortex Dipole

A vortex dipole is made of two vortices of opposite sign. Figure 7.6, shows the streamlines around the coupled vortices in a frame moving with the dipole. Each vortex is moved by the velocity field generated by its partner vortex. In this case we find,

$$\Omega = \Gamma\delta(x - Ut)\delta(y - \frac{d}{2}) - \Gamma\delta(x - Ut)\delta(y + \frac{d}{2}).$$

Thus,

$$U = \frac{\Gamma}{2\pi d}.$$

7.2.4 A Set of Point Vortices

The ideas behind the previous example can be generalized. Consider a special class of solutions to the 2D vorticity equation, such that the vorticity Ω is concentrated in a set of point vortices. Each of the point vortices will be moved by the velocity field produced by all the other point vortices, $\mathbf{u}_i$. We find, for N point vortices, that

$$\Omega = \sum_{i=1}^{N} \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i(t)).$$

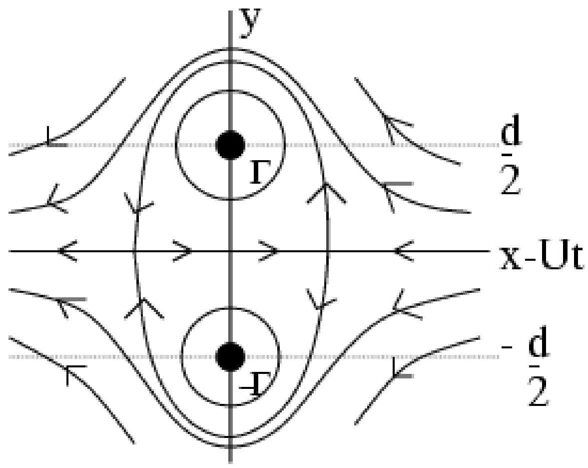

Figure 7.6: Vortex Dipole

$$\frac{d}{dt}\mathbf{x}_i = \mathbf{u}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \Gamma_j \frac{\mathbf{e}_z \times (\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^2}$$

7.3 Exercises

Find the streamfunction ψ for the examples above.

7.3.1 Shear Flow

$$u = \partial_y \psi \Rightarrow \psi = \int f(y)dy.$$

The constant in ψ can be chosen arbitrarily, that is, it won't change $\mathbf{u}$.

7.3.2 Circular Vortex

$$u_\theta = -\partial_r \psi \Rightarrow \psi(r) = -\int g(r)dr.$$

Rankine Vortex

$$\psi = \begin{cases} -Cr^2/2 & r < a \\ -Ca^2 ln(\frac{r}{a}) & r \geq a. \end{cases}$$

The constant is chosen so that ψ is continuous at $r = a$

Point Vortex

$$\psi = -\frac{\Gamma}{2\pi} ln(r).$$

7.3.3 Vortex Dipole

$$\psi = -\frac{\Gamma}{2\pi} ln \left| (x-Ut)^2 + (y-\frac{d}{2})^2 \right|^{1/2} + \frac{\Gamma}{2\pi} ln \left| (x-Ut)^2 + (y+\frac{d}{2})^2 \right|^{1/2}.$$

This expression is obtained by using the above ψ for a point vortex at each of the point vortices.

7.3.4 Set of Point Vortices

Again using the streamfunction generated by a single point vortex we find,

$$\psi(\mathbf{x}) = -\sum \frac{\Gamma_i}{2\pi} ln|\mathbf{x} - \mathbf{x}_i|$$