Lecture 9

2D Irrotational Motion

9.1 Velocity Potential and Circulation

For an incompressible inviscid fluid, we can introduce a velocity potential $\phi(\mathbf{x}, t)$ such that the irrotational velocity field is,

$$\mathbf{u} = \nabla\phi. \tag{9.1}$$

One notices the condition $\nabla \times \mathbf{u} = 0$ is satisfied automatically. In a simply connected domain, figure 9.1, ϕ is a single valued function. Solving for ϕ we find,

$$\phi(\mathbf{x}, t) = \int_0^{\mathbf{x}} \mathbf{u}(\mathbf{x}') \cdot d\mathbf{x}',$$

where integration is performed over an arbitrary curve connecting 0 and x. (*Question*: Why is $\phi(\mathbf{x})$) independent of the curve for simply connected domains?). This represen-

Figure 9.1: Simply Connected Domain

Figure 9.2: Multi-Connected Domain

tation is valid also in multi-connected fluid regions, figure 9.2, ϕ is multi-valued in this case.

We can relate a velocity *circulation* Γ to the potential ϕ as follows,

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x} = \oint_C \nabla\phi \cdot d\mathbf{x} = [\phi]_C \, ,$$

where the last expression denotes the change in ϕ (if any) after one circuit round C. The circulation is zero in a simply connected case, but can be finite in multi-connected regions, as in figure 9.3.

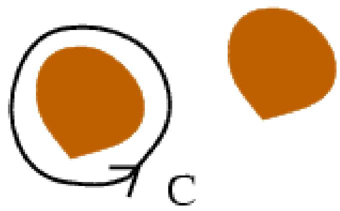

Figure 9.3: Multi-Connected Domain

9.2 Example Potentials

9.2.1 A Uniform Flow

A uniform flow, of velocity U, has a potential $\boxed{\phi = Ux}$.

9.2.2 A Pure-Strain Flow

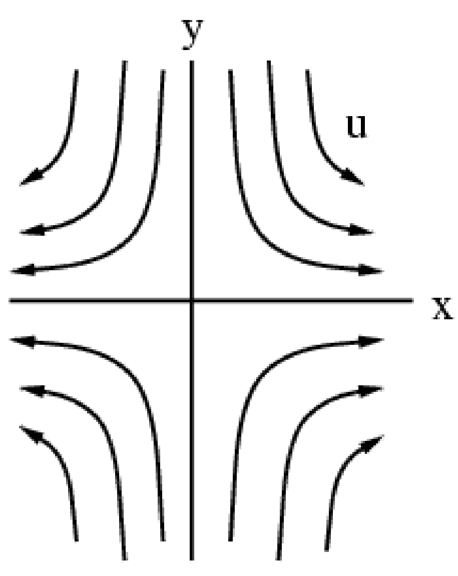

Figure 9.4: A Pure-Strain Flow

A pure strain flow, figure 9.4, has a velocity profile like

$$u = \alpha x,$$
$$v = -\alpha y,$$
$$w = 0,$$

where α is a constant. This produces a potential,

$$\boxed{\phi = \tfrac{1}{2}\alpha(x^2 - y^2)}.$$

9.2.3 A Point Vortex

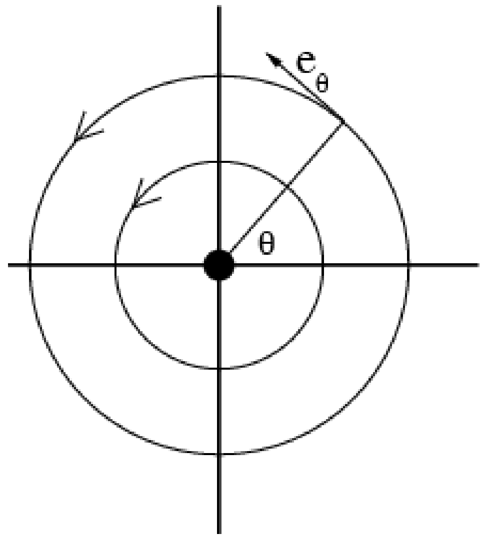

Figure 9.5: A Point Vortex

Recall, a point vortex, figure 9.5, has a velocity profile of,

$$\mathbf{u} = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta,$$

where $\mathbf{e}_\theta$ is a unit tangential vector. However, we should be aware that the domain in this case is multi-connected because the point $r = 0$ is excluded. We find the potential in this case is,

$$\boxed{\phi = \tfrac{\Gamma}{2\pi}\theta},$$

with the circulation Γ being,

$$\Gamma = \begin{cases} \Gamma & \text{for } C\text{'s encompass } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

9.3 Streamfunction and Complex Potential

As you have already seen, we can introduce a streamfunction ψ to describe a 2D incompressible inviscid flow (it doesn't have to be necessarily irrotational), such that

$$u = \partial_y \psi, \tag{9.2}$$

$$v = -\partial_x \psi. \tag{9.3}$$

Then $\nabla \cdot \mathbf{u}$ is satisfied automatically.

Note: $(\mathbf{u} \cdot \nabla)\psi = 0$, i.e. ψ is constant for any streamline of a stationary flow.

If the fluid flow is *also* irrotational, we can also introduce our velocity potential (9.1). The two are related in the following way,

$$u = \partial_x \phi = \partial_y \psi, \quad (9.4)$$
$$v = \partial_y \phi = -\partial_x \psi.$$

However, these are just the *Cauchy-Riemann* equations for a complex function of the form $\boxed{\chi = \phi + i\psi}$. χ is called a *Complex Potential.* If the derivatives of ϕ and ψ are continuous then the relations (9.4) mean that χ is an analytic function. This, in turn, implies that both ϕ ans ψ satisfy the Laplace equation.

$$\nabla \cdot \mathbf{u} = 0 \Rightarrow \nabla^2 \phi = 0,$$
$$\nabla \times \mathbf{u} = 0 \Rightarrow \nabla^2 \psi = 0.$$

Further, $\partial_z \chi$ gives us the velocity components,

$$\partial_z \chi = \partial_x \phi + i \partial_x \psi = u - iv.$$

9.4 Complex Potential Examples

9.4.1 Uniform Flow at an Angle α

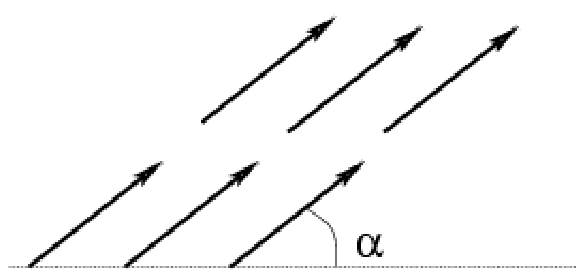

Figure 9.6: Uniform Flow at an angle α

For a uniform flow at an angle α, figure 9.6 the velocity profile is

$$u = U \cos \alpha,$$
$$v = U \sin \alpha.$$

Therefore,

$$\partial_z \chi = U e^{-i\alpha} \Rightarrow \chi = U z e^{-i\alpha}.$$

9.4.2 Point Vortex

Recall, for a point vortex, we already know

$$\mathbf{u} = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta,$$
$$\phi = \frac{\Gamma\theta}{2\pi}.$$

Now, it is useful to re-write the velocity in terms of ψ, via $\boxed{\mathbf{u} = \nabla \times (\psi \mathbf{e}_z)}$. (You should check this). This gives us a quick way of writing the radial component, as follows

$$u_r = \frac{1}{r}\partial_\theta \psi,$$

and for the azimuthal component,

$$u_\theta = -\partial_r \psi = \frac{\Gamma}{2\pi r}.$$

Therefore,

$$\psi = -\frac{\Gamma}{2\pi}\ln r.$$

So,

$$\begin{aligned} \chi &= \phi + i\psi \\ &= \frac{\Gamma}{2\pi}(\theta - i\ln r) \\ &= -\frac{i\Gamma}{2\pi}(\ln r + i\theta) \\ &= -\frac{i\Gamma}{2\pi}\ln z. \end{aligned}$$

9.4.3 Do it yourself!

Use a Taylor series to show that near a stagnation point,

$$\chi \cong \frac{1}{2}\alpha(z - z_0)^2.$$

(A stagnation point is a point z_0 where $\mathbf{u} = 0$). By rotation of the complex plane make α real. By shifting, get rid of z_0. Deduce that the resulting flow coincides with the pure-strain flow.

Next lecture will be devoted entirely to another important example.